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Parahoric projection for twin trees

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Parahoric projection for twin trees

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À papa et maman. Merci pour tout.

Summary

Geometrical configurations (for example, of points and lines in a plane) have been studied for a long time. The simplest configurations are often rigid, and so we search for more interesting ones. Calderbank and Noppakaew [16] introduced the idea of *parabolic projection* in order to obtain new geometrical configurations from spherical buildings. Roughly speaking, a building, introduced by Tits [41] is a chamber system with some added constraints. There are three types of buildings: spherical, affine and indefinite.

The motivation for this thesis is to start developing a theory of *parahoric projection*, the analogue of parabolic projection, but for the case of affine buildings since it is the next natural step after the spherical case.

In the finite parabolic projection one key ingredient is the notion of oppositeness since the idea is to project so-called weakly opposite apartments. In this thesis, we define the analogue of this in the affine case using the notion of twin buildings, and we show that under some constraints these weakly opposite apartments exist and that the lifting problem has a solution in the case of $\widetilde{A_1}$.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Geometrical configurations

The study of geometrical configurations date back to the XIX Century and was later popularised by Hilbert and Cohn-Vossen in their book "Geometry And The Imagination" [23].

In classical plane geometry, a geometrical configuration is a collection of points and lines with prescribed incidences between them. We say that a point and a line are incident if the point is contained in the line. An example of this is the complete quadrangle (shown in Figures 1-1 below) where it is a collection of four points in the plane, no three of which are on a common line, and six lines, each one incident to exactly two of these points.

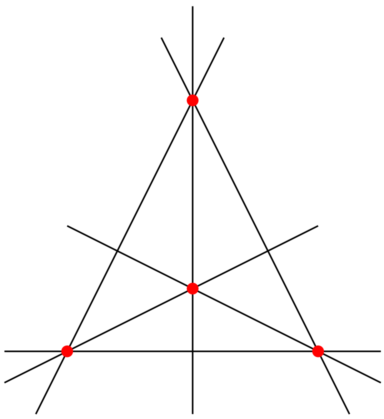

Figure 1-1: *Example of a geometrical configuration: a complete quadrangle*

A geometrical configuration is a morphism of incidence systems. In the previous example, the codomain is the incidence of points and lines on a projective plane, and the domain the incidence structure of the tetrahedron.

We formalise this by saying that there are two aspects of geometrical configurations:

(1) The geometrical space where the configuration lies (which will be the codomain of the morphism),

(2) The abstract incidence structure (which will be the domain).

An interesting and large class of examples of geometrical configurations is when the domain is a polytope. In this case, the polytope has an incidence structure: the structure with its vertices, edges, faces, etc. As before incidence is defined by inclusion. A particular class of such configurations is the generalised Clifford-Cox configurations. Longuet-Higgins [30] showed that these configurations have Coxeter polytopes as their incidence structure and the relations between these polytopes can be used to construct the configurations inductively.

A key observation is that projective space is a generalised flag variety. Generalised flag varieties are projectivized highest weight orbits in representations of a semisimple algebraic group. This is an interesting observation since semisimple algebraic groups have Coxeter groups associated to them, namely their Weyl groups.

If we choose a Cartan subalgebra of the associated Lie algebra (of this algebraic group) then the weight spaces of this subalgebra form a geometrical configuration inside the flag variety. Examples of this are the triangle in the projective plane, or the simplex in projective space more generally.

This geometrical approach to semisimple algebraic groups was introduced by Jacques Tits. These configurations are so-called apartments in a building.

This was all done in the finite case, i.e. when the Coxeter group is finite, and the buildings associated to semisimple algebraic groups are spherical.

In the context of classical geometrical configurations, it is natural and useful to restrict to the finite case. Indeed, Longuet-Higgins only considered finite Coxeter groups and did not consider the infinite case to be valid. The next simplest case after the finite case is the affine case, where the polytope is now a tiling of the Euclidean space.

1.2 Parabolic projection

Configurations associated to apartments tend to be too simple or rigid, so in order to create and study more complicated and interesting ones, Calderbank and Noppakaew in [16] introduced the notion of *parabolic projection*. A simple example of this is the projection $\pi : \mathbb{P}^3 \mapsto \mathbb{P}^2$ away from a point. In this example, an apartment is a tetrahedron, and so if we project from a point q which is not on the tetrahedron, we get a quadrangle.

In this thesis we begin investigating configurations where the domain is an infinite Coxeter group. One motivation for this is the growing interest in discrete integrable geometry, for example discrete conjugate nets, also known as Q-nets, and discrete line congruences, which may be viewed as infinite configurations of points and lines, and provide examples of integrable systems.

We would like to construct these infinite geometrical configurations using affine buildings. Affine buildings are in plentiful supply; for example, there are the Bruhat-Tits buildings associated to semisimple algebraic groups over a field with a valuation. These buildings may be viewed as generalisations of parabolic buildings (which are spherical) in which a finite dimensional Lie algebra is replaced by a loop algebra, or an affine Lie algebra.

However, in order to talk about 'opposite-ness' (in the affine case) we need to use the notion of 'twin buildings'. Twin buildings are a pair of affine buildings with an 'opposite-ness' relation between them. They arise from Bruhat-Tits buildings when the field has two valuations. The prototypical example is the field of rational functions where the two valuations are the order of the function at 0 and at ∞.

In this thesis we prove existence of weakly opposite apartments for for the simplest class of affine buildings, those of type $\widetilde{A_1}$, which are bipartite trees. Twin buildings of type $\widetilde{A_1}$ have been studied by Ronan and Tits [35] who called them twin trees.

Our main results are in Chapter 5. We solve the lifting problem and show existence of weakly opposite apartments. Furthermore, we construct an apartment in the building of type $\widetilde{A_1}$ over a rational function field $\mathbb{F}(t)$ with valuation v_0, which is weakly opposite to the standard parahoric which is the stabiliser of the class of standard lattice $\mathbb{F}[t^{-1}]^2$. Finally, we parahorically project it onto a geometrical configuration.

CHAPTER 2

INCIDENCE SYSTEMS, CHAMBER SYSTEMS AND BUILDINGS

In this thesis the term "graph" will always refer to an undirected graph with no loops and no multiple edges. If Γ is a graph then we will denote by $|\Gamma|$ the vertex set of Γ and if there is no ambiguity we will denote it by Γ instead. The set of edges of Γ (which are two-element subsets of $|\Gamma|$) will be denoted by E_Γ.

Throughout this chapter, the set I will be fixed and finite.

The approach we take in this chapter mainly follows [45], [15] and [16].

2.1 Incidence systems

Definition 2.1.1 (Incidence system and type function)**.** An *incidence system* over I is a symmetric reflexive relation, in other words it is a graph Γ equipped with a surjective map $t : \Gamma \to I$ such that for all edges a—$b \in E_\Gamma$, $t(a) \neq t(b)$. We call the map t a *type function*.

For any edge a—b, we say that the vertices a and b are *incident* in Γ. Note that the relation being reflexive implies that every vertex in Γ is incident to itself.

Definition 2.1.2 (Incidence morphism)**.** Let Γ_1 and Γ_2 be two incidence systems over I. A map $\Phi : \Gamma_1 \to \Gamma_2$ is an *incidence morphism* if it is a type-preserving graph morphism.

We denote the set of automorphisms of Γ by $Aut(\Gamma)$.

Definition 2.1.3 (Flag)**.** Let Γ be an incidence system. A set $\mathcal{F}$ of mutually incident elements is called a *flag* of Γ. It is of *type* $t(\mathcal{F})$ and of *rank* $|t(\mathcal{F})|$.

A flag is *maximal* if it is not contained in a larger flag. A *full flag* is a flag of type I.

2.2 Chamber systems

Definition 2.2.1 (Chamber system)**.** A *chamber system* over I is a graph Δ with an edge labelling $\lambda\colon E_\Delta \to I$ such that for each $i \in I$, the *i-adjacency* relation $x \overset{i}{—} y$ (i.e. $x = y$ or there exists an edge with label i joining x and y) is an equivalence relation.

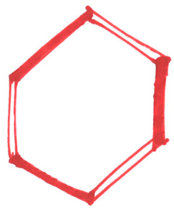

Figure 2-1: *Example of a chamber system Δ over $I = \{\circ, \bullet\}$*

We call the elements of $|\Delta|$ the *chambers* of Δ and the equivalence class of any chamber $x \in |\Delta|$ under i-adjacency is called its *i-panel*.

A *chamber morphism* $\Phi : \Delta \to \Delta'$ is simply a map on vertices such that $\Phi(x) \overset{i}{—} \Phi(y)$ if $x \overset{i}{—} y$. It is a *chamber isomorphism* if it satisfies: $\Phi(x) \overset{i}{—} \Phi(y)$ if and only if $x \overset{i}{—} y$ for all $x, y \in \Delta$.

Let us denote the set of *automorphisms* of Δ, i.e. the chamber isomorphisms $\Delta \to \Delta$, by $Aut(\Delta)$.

The following definition will enable us to define the concept of a chamber morphism between chamber systems that are not necessarily over the same set.

Definition 2.2.2 (Pull back chamber system)**.** Let I and I' be two sets. Let Δ be a chamber system over I and let $\nu : I' \to I$ be a map. Then the *pull back chamber system* of Δ over I' induced by ν is denoted $\nu^*\Delta$ and is the chamber system over I' such that $|\nu^*\Delta| = \{x \in \Delta | \exists y \in |\Delta| \text{ and } i \in I \text{ such that } x \overset{\nu(i)}{—} y\}$ and for all $i \in I'$, $x \overset{i}{—} y$ in $\nu^*\Delta$ if $x \overset{\nu(i)}{—} y$ in Δ.

If ν is the inclusion map $I \subseteq I'$ then $\nu^*\Delta$ is a subgraph of Δ.

Definition 2.2.3 (Chamber morphism over $\nu : I' \to I$)**.** Let Δ and Δ' be chamber systems over I and I' respectively. We say that the map $\Phi : \Delta \to \Delta'$ is a *chamber morphism over some* $\nu : I' \to I$ if it induces a chamber morphism $\Phi : \nu^*\Delta \to \Delta'$.

Definition 2.2.4 (Path, geodesic)**.** Let Δ be a chamber system and let $x, y \in \Delta$. A *path* $\pi : x \to y$ of length k is a sequence π of chambers $x = u_0, u_1, \ldots, u_k = y$ such that $\{u_j, u_{j+1}\}$ is an edge of Δ for all $0 \leqslant j < k$. We say that π is *geodesic* from x to y if its length is minimal.

Some authors call geodesics "reduced paths". Using the same notations as the ones in the definition above, we say that $\pi : x \to y$ is a *path of type* $f = i_1 i_2 \ldots i_k$ where $u_{j-1} \overset{i_j}{—} u_j$ for all $0 < j \leqslant k$.

A chamber system Δ is *connected* if for every pair of chambers in Δ there exists a path joining them. A connected chamber system Δ over I is *thin* if every vertex is an endpoint of exactly one edge with each label $j \in I$.

Let Δ be a thin chamber system and let $i \in I$. Then i defines a fixed point free involution of $|\Delta|$ sending x to y if and only if $x \overset{i}{—} y$ (for $x \neq y$). We write $y = xi$ (and hence $x = yi$).

Definition 2.2.5 (Structure group)**.** Let Δ be a thin chamber system. The subgroup of $\mathrm{Sym}(|\Delta|)$ generated by the involutions $c \mapsto ci$ for all $i \in I$ is called the *structure group* of Δ, and is denoted by W_Δ.

If Δ is a thin chamber system with structure group W_Δ then we say that Δ is of type W_Δ.

Definition 2.2.6 (Convex subgraph)**.** Let Δ be a chamber system and let Δ' be a subgraph of Δ. Then Δ' is convex if for all chambers x and y of Δ' every geodesic in Δ from x to y is also contained in Δ'.

Definition 2.2.7 (Homogeneous thin chamber system)**.** Let Δ be a thin chamber system. We say that Δ is *homogeneous* if $Aut(\Delta)$ acts transitively on the chambers of Δ, i.e. for all chambers $x, y \in |\Delta|$ there exists $\phi \in Aut(\Delta)$ such that $y = \phi(x)$.

Proposition 2.2.8 (See for example [16, page 20])**.** *Let Δ be a homogeneous connected thin chamber system. Then W_Δ acts both freely and transitively on Δ.*

Therefore in a homogeneous connected thin chamber system Δ we can define the map $\delta_\Delta : \Delta \times \Delta \to W_\Delta$ such that $\delta_\Delta(x, y) = w$ if and only if $y = xw$.

2.3 Coxeter chamber systems

We begin with some basic notions and definitions about Coxeter groups. There are three types of Coxeter groups: finite, affine and indefinite. In what follows we focus on the first two types.

Definition 2.3.1 (Coxeter matrix)**.** A *Coxeter matrix* over a set I is a symmetric matrix $m\colon I \times I \to \mathbb{Z}^+ \cup \{\infty\}$ with $m_{ij} = 1$ if and only if $i = j$.

Definition 2.3.2 (Coxeter group)**.** Let m be a Coxeter matrix. The *Coxeter group* W associated to m is

$$W := \langle i \in I | (ij)^{m_{ij}} = 1 \text{ for } m_{ij} \neq \infty \rangle.$$

The pair (W, I) is called a *Coxeter system.* We say that $f = i_1 i_2 \cdots i_k \in W$ is a *reduced word* if there is no shorter expression of it as a product of the generators in I, in which case we say that k is the *length* of f.

Definition 2.3.3 (Coxeter diagram)**.** Let m be a Coxeter matrix over the set I. The labelled graph Π with vertex set I and the edge set consisting of the unordered $\{i, j\}$ such that $m_{ij} \geqslant 3$, including ∞, is called the *Coxeter diagram* associated to m (and hence to the Coxeter group W). The edge $\{i, j\}$ is labelled by m_{ij}.

Theorem 2.3.4 ([10, page 193])**.** *The list of Coxeter diagrams in Figure 2-2 represents all the possible connected diagrams of finite Coxeter groups.*

Coxeter groups that are not finite but that contain a normal abelian subgroup such that the corresponding quotient group is finite are called *affine Coxeter groups.* The Coxeter diagrams in Figure 2-3 are called *irreducible affine Coxeter diagrams.*

Note that any Coxeter diagram of type X_n (where X is A, B, C, D, E, F or G) in Figure 2-2 can be obtained from the Coxeter diagram $\widetilde{X_n}$ in Figure 2-3 by deleting a single vertex as well as the edge (or two edges in the case of $\widetilde{A_n}$) and the label if any connected to it. Such a vertex is called *special.* For example, by symmetry, all vertices in the $\widetilde{A}_n$ graph are special.

We are now ready to introduce the concept of a Coxeter chamber system.

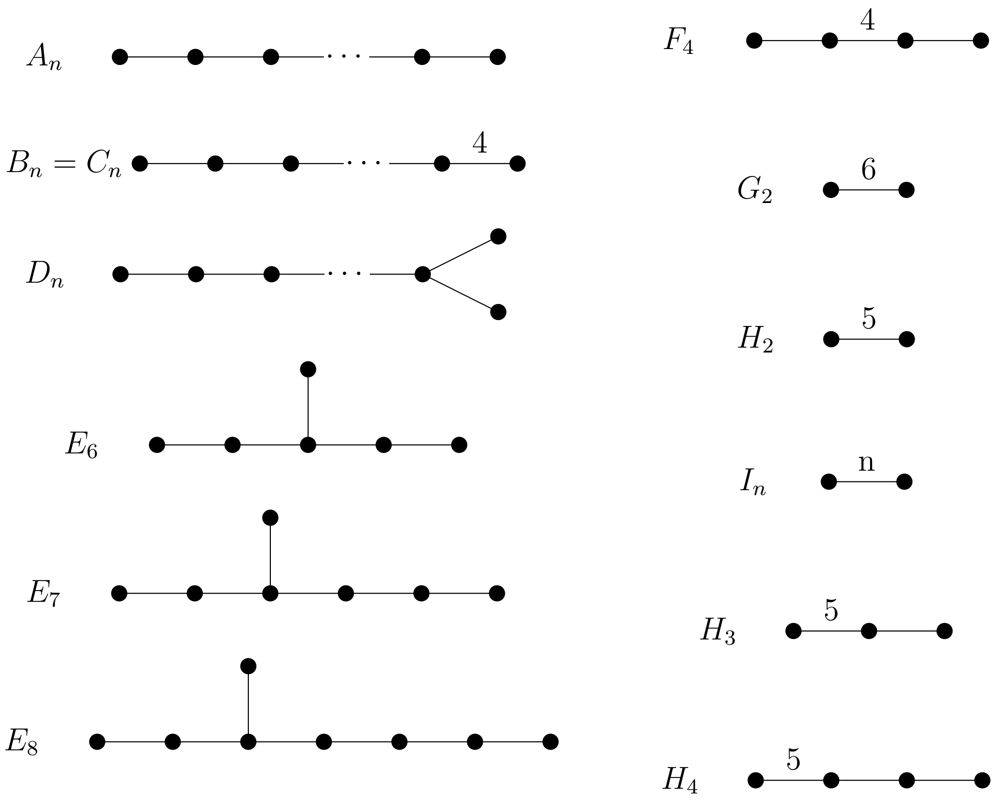

Figure 2-2: *Connected Coxeter diagrams of the finite Coxeter groups*

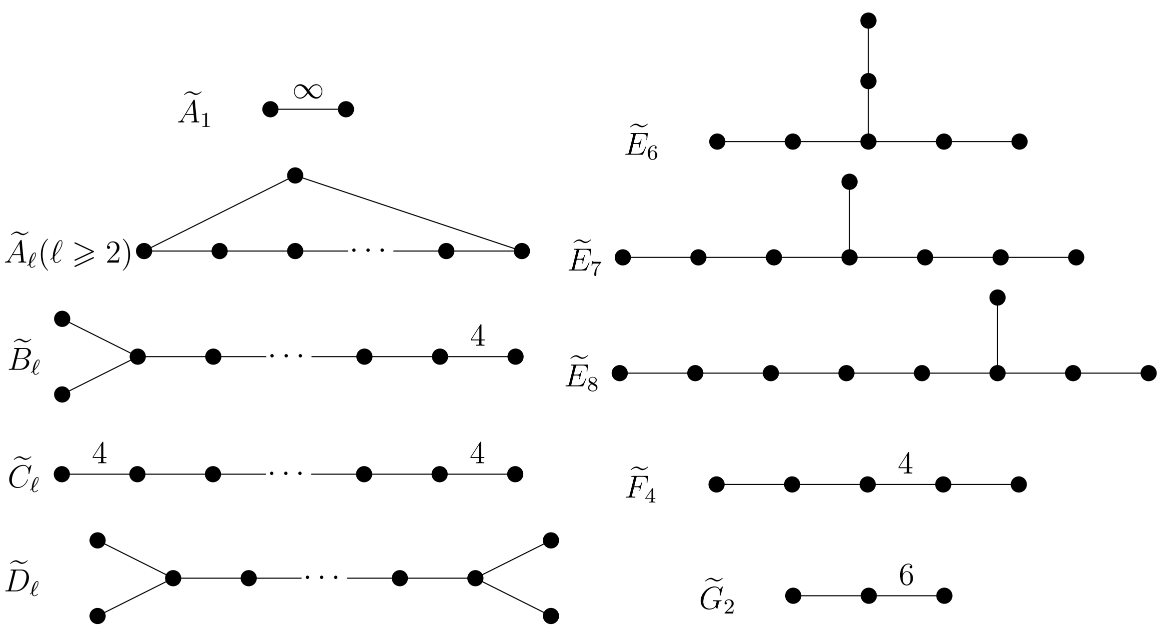

Figure 2-3: *The irreducible affine Coxeter diagrams*

Definition 2.3.5 (Coxeter chamber system)**.** Let W be a Coxeter group with Coxeter diagram Π (that has a vertex set I). A *Coxeter chamber system* over I of type Π is a homogeneous thin chamber system over I with structure group W.

Proposition 2.3.6 ([45])**.** *Let (W, I) be a Coxeter system with Coxeter diagram Π. Let Σ_Π be a chamber system over I with vertex set W and such that $x \overset{i}{—} y$ if and only if $y = xi$, for $x, y \in |\Sigma_\Pi|$ and $i \in I$. Then any chamber system Σ over I is Coxeter of type Π if and only if it is isomorphic to Σ_Π.*

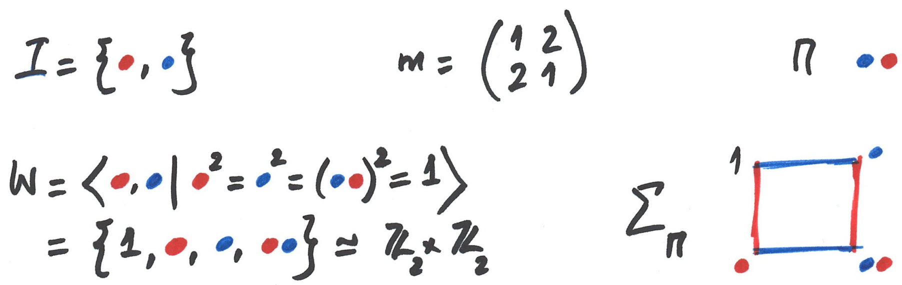

Figure 2-4: *Example of a Coxeter chamber system Σ_Π where Π is of type $A_1 \times A_1$*

Note that this is equivalent to saying that Σ_Π is the Cayley graph of (W, I).

2.4 Buildings

Historically, buildings were defined either using simplicial complexes with subcomplexes called "apartments" or with a distance function on chambers. The modern approach is via chamber systems. This can be done in three different ways: using a distance function; using a distance function and apartments or by using apartment complexes. As we will see, these three definitions are equivalent.

In this section, the chamber system Δ will always be connected.

Definition 2.4.1 (Apartment)**.** Let Δ be a chamber system. Any subgraph of Δ that is a Coxeter chamber system is called an *apartment* of Δ.

It follows that apartments are homogeneous thin chamber systems and hence any apartment A of type Π has an associated distance function $\delta_A : A \times A \to W$ such that $\delta_A(x, y) = w$ if and only if $y = xw$ (where W is the associated Coxeter group of Π).

Definition 2.4.2 (Compatible apartment)**.** Let Δ be a chamber system and let W be a Coxeter group with Coxeter diagram Π. Let $\delta_\Delta : \Delta \times \Delta \to W$ be a map and let A be an apartment in Δ of type Π. We say that A is *compatible* with δ_Δ if $\delta_\Delta(x, y) = \delta_A(x, y)$ for all $x, y \in A$.

Definition 2.4.3 (Apartment complex)**.** An *apartment complex* $\mathcal{A}$ in a chamber system Δ is a set of apartments of Δ satisfying the following two conditions

(1) for any chambers $x, y \in \Delta$, there exists an apartment $A \in \mathcal{A}$ such that $x, y \in A$;

(2) for any apartments $A_1, A_2 \in \mathcal{A}$ and any chambers $x, y \in A_1 \cap A_2$, there is a chamber isomorphism $A_1 \to A_2$ fixing x and y.

Those conditions in the previous definition imply that the apartments in an apartment complex are all of same type Π. Let W be the Coxeter group associated to the diagram Π. We can define the map $\delta_{\mathcal{A}} : \Delta \times \Delta \to W$, such that $\delta_{\mathcal{A}_{|A}} = \delta_A$ for all $A \in \mathcal{A}$. This map is well-defined: the first condition of the previous definition implies that it is everywhere defined and the second condition implies that it is uniquely defined.

Theorem 2.4.4. *Let Δ be a chamber system. Let W be a Coxeter group with Coxeter diagram Π and let $\delta : \Delta \times \Delta \to W$ be a map such that $\delta(x, y) = \delta(y, x)^{-1}$ and such that if $x, x' \in \Delta$ are i-adjacent then $\delta(x', y) \in \{\delta(x, y), i\delta(x, y)\}$. The following conditions are equivalent.*

(1) *$\delta(x, y) = f$ if and only if there exists a geodesic in Δ of type f from x to y, for all $x, y \in \Delta$.*

(2) *Any two chambers x, y in Δ are contained in a common apartment that is compatible with δ.*

(3) *There is an apartment complex $\mathcal{A}$ in Δ such that $\delta = \delta_{\mathcal{A}}$.*

Sketch of Proof. (3) $\Longrightarrow$ (1). This follows from the fact that apartments are convex. (1) $\Longrightarrow$ (2). Here we need to construct apartments. See [36] for a detailed proof. (2) $\Longrightarrow$ (3). Let $\mathcal{A}$ denote the set of all apartments compatible with δ. Condition (1) of Definition 2.4.3 is straightforward and condition (2) is implied from compatibility with δ. □

If a chamber system Δ together with a distance function δ satisfies any of the conditions in the previous theorem, it is called a *building of type* Π (where Π is the Coxeter diagram of W). A chamber system Δ together with an apartment complex $\mathcal{A}$ implies the existence of a distance function δ on Δ such that (Δ, δ) is a building. In some cases, and by abuse of notation, we will denote this building by $(\Delta, \mathcal{A})$; this is done because in the case of parabolic buildings for example, it is easier to define a specific apartment complex than a distance function on Δ.

Theorem 2.4.5 ([1])**.** *Let* $(\Delta, \mathcal{A})$ *be a building and let* A *be a compatible apartment in* Δ*. Then* $A \in \mathcal{A}$*.*

CHAPTER 3
BUILDINGS: SPHERICAL AND AFFINE

Buildings fall into three categories: spherical, affine and indefinite. In this thesis we discuss the first two types, which have been extensively studied and classified. In this chapter we focus on the spherical buildings and an important subset of them: parabolic buildings.

3.1 Spherical buildings and parabolic buildings

Definition 3.1.1 (Spherical building)**.** Let (Δ, δ) be a building of type Π. We say that it is a *spherical building* if the Coxeter group W with diagram Π is finite.

An important class of spherical buildings is the one of the so-called 'parabolic buildings'. We can define parabolic buildings associated to either the projective space $\mathbb{P}^n$, to the Lie group $\mathrm{SL}_{n+1}(\mathbb{F})$ or to the Lie algebra $\mathfrak{sl}_{n+1}(\mathbb{F})$, as we will see below.

Projective geometry

Let $\mathbb{F}$ be any field and let $\mathrm{SL}_{n+1}(\mathbb{F})$ denote the *special linear group* of degree $n+1$ over $\mathbb{F}$ consisting of $(n+1)\times(n+1)$ matrices with determinant 1.

We begin by defining the incidence system $\mathcal{I}_{\mathbb{P}^n}$. As a set, define:

$$\mathcal{I}_{\mathbb{P}^n} := \{\mathbb{P}(W), 0 < W < \mathbb{F}^{n+1}\}.$$

Now note that $\mathrm{SL}_{n+1}(\mathbb{F})$ acts on $\mathcal{I}_{\mathbb{P}^n}$ as follows: $g.\mathbb{P}(W) := \mathbb{P}(g.W)$, where $g.W := \{gw | w \in W\}$. Let S be the set of orbits that we (bijectively) label by the dimension of W and so we define the type function $t_{\mathcal{I}} : \mathcal{I}_{\mathbb{P}^n} \to S$ sending every element in $\mathcal{I}_{\mathbb{P}^n}$ to its corresponding orbit. The incidence relation is inclusion, i.e. $\mathbb{P}(W)$ and $\mathbb{P}(V)$ in $\mathcal{I}_{\mathbb{P}^n}$ are incident if $\mathbb{P}(W) \subseteq \mathbb{P}(V)$ or $\mathbb{P}(V) \subseteq \mathbb{P}(W)$.

Incidence systems associated to the apartments: pick a basis of $\mathbb{F}^{n+1}$. Then the projectivization of the subspaces generated by the span of the non-empty proper subsets of this basis, together with the adequate incidences (as described above) form an apartment in $\mathcal{I}_{\mathbb{P}^n}$.

If we look at $\mathbb{P}^2$ as an example, a chamber is a full flag, i.e. it is a projective point on a projective line. The Coxeter group is the finite group Sym_3 with Coxeter diagram of type A_2 and the apartments are as illustrated in Figure 3-1 below.

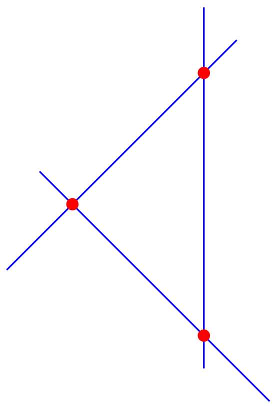

Figure 3-1: *Incidence system of an apartment in* $\mathbb{P}^2$

Finally, the following result is standard, see for example [16] and [37] for a detailed proof.

Theorem 3.1.2. *The chamber system associated to $\mathcal{I}_{\mathbb{P}^n}$ is a spherical building of type A_n.*

Lie groups

Let $\mathrm{Stab}_{\mathrm{SL_{n+1}(\mathbb{F})}}(\mathbb{P}(W)) := \{g \in \mathrm{SL}_{n+1}(\mathbb{F}) | g.W = W\}$, for all $\mathbb{P}(W) \in \mathcal{I}_{(\mathrm{SL}_{n+1},\mathbb{F})}$. Let us define the incidence system $\mathcal{I}_{(\mathrm{SL}_{n+1},\mathbb{F})}$ as follows. As a set, let

$$\mathcal{I}_{(\mathrm{SL}_{n+1},\mathbb{F})} := \{\mathrm{Stab}_{\mathrm{SL_{n+1}(\mathbb{F})}}(\mathbb{P}(W)) | \mathbb{P}(W) \in \mathcal{I}_{\mathbb{P}^n}\}.$$

The types of $\mathcal{I}_{(\mathrm{SL}_{n+1},\mathbb{F})}$ are the orbits when letting $\mathrm{SL}_{n+1}(\mathbb{F})$ act on $\mathcal{I}_{(\mathrm{SL}_{n+1},\mathbb{F})}$: $h.\,\mathrm{Stab}_{\mathrm{SL_{n+1}(\mathbb{F})}}(P) = h\,\mathrm{Stab}_{\mathrm{SL_{n+1}(\mathbb{F})}}(P)h^{-1}$ for all $h \in \mathrm{SL}_{n+1}(\mathbb{F})$ and $\mathrm{Stab}_{\mathrm{SL_{n+1}(\mathbb{F})}}(P) \in$

$\mathcal{I}_{(\mathrm{SL}_{n+1},\mathbb{F})}$. Now note that $P_2 = g.P_1$ implies that $\mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{F})}(P_2) = g.\,\mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{F})}(P_1)$. Therefore we can identify the types of $\mathcal{I}_{(\mathrm{SL}_{n+1},\mathbb{F})}$ with the types of $\mathcal{I}_{\mathbb{P}^n}$.

Finally, we need to define the incidence relation. We say that a subgroup of $\mathrm{SL}_{n+1}(\mathbb{F})$ is *parabolic* if it contains the stabiliser of a full flag in $\mathcal{I}_{\mathbb{P}^n}$. Now let $\mathrm{Stab}_{\mathrm{SL_{n+1}}(\mathbb{F})}(P_1)$ and $\mathrm{Stab}_{\mathrm{SL_{n+1}}(\mathbb{F})}(P_2)$ be in $\mathcal{I}_{(\mathrm{SL}_{n+1},\mathbb{F})}$: we say that they are incident if their intersection is parabolic.

Theorem 3.1.3. *The map $P \mapsto \mathrm{Stab}_{\mathrm{SL_{n+1}}(\mathbb{F})}(P)$ is an incidence system isomorphism $\mathcal{I}_{\mathbb{P}^n} \to \mathcal{I}_{(\mathrm{SL}_{n+1},\mathbb{F})}$.*

Proof. Call this map ϕ. It is clearly well-defined and type preserving. Let P_1 and P_2 be incident in $\mathcal{I}_{\mathbb{P}^n}$, then we can make P_1 and P_2 into a full flag by adding the necessary incident projective spaces as follows. Let $P_1 = \mathbb{P}(W_1)$ and $\mathbb{P}(W_2)$ with $\dim W_1 = n_1$ and $\dim W_2 = n_2$ and without loss of generality let us assume that $P_1 \subseteq P_2$, i.e. $W_1 \subseteq W_2$. Pick a basis $x_1, \dots, x_{n_1}$ of W_1 and complete it into a basis $x_1, \dots, x_{n_1}, \dots, x_{n_2}$ of W_2 and then into a basis $x_1, \dots, x_n, x_{n+1}$ of $\mathbb{F}^{n+1}$. Let F_1 be the subspace generated by x_1, F_2 the subspace generated by x_1, x_2 and generally let F_i be the subspace generated by the first i vectors in this basis. This means that $W_1 = F_{n_1}$ and $W_2 = F_{n_2}$. Finally note that $\mathcal{F} := \{F_1, F_2, \dots, F_n\}$ is a full flag.

Now the stabilisers of $\mathcal{F}$, by construction, must stabilise both P_1 and P_2, and so $\mathrm{Stab}_{\mathrm{SL_{n+1}}(\mathbb{F})}(P_1) \cap \mathrm{Stab}_{\mathrm{SL_{n+1}}(\mathbb{F})}(P_2)$ is parabolic, i.e. $\mathrm{Stab}_{\mathrm{SL_{n+1}}(\mathbb{F})}(P_1)$ and $\mathrm{Stab}_{\mathrm{SL_{n+1}}(\mathbb{F})}(P_2)$ are incident in $\mathcal{I}_{(\mathrm{SL}_{n+1},\mathbb{F})}$.

The map ϕ is by construction a surjective map, and so we only need to show that it is injective. Let $P_1 = \mathbb{P}(W_1)$ and $P_2 = \mathbb{P}(W_2)$ and without loss of generality, let $\dim W_1 = n_1 \leqslant \dim W_2 = n_2 < n+1$. Suppose $P_1 \neq P_2$. Pick a basis of W_1: $x_1, x_2, \dots, x_{n_1}$. Now since $W_1 \neq W_2$, there exists an element $y \in W_2$ such that $y \notin W_1$ that is linearly independent from $x_1, \dots, x_{n_1}$. And since W_1 and W_2 are proper subspaces of $\mathbb{F}^{n+1}$, there exists also an element $\tilde{y} \notin W_2$ that is also linearly independent from $x_1, \dots, x_{n_1}$.

Finally, a well known projective geometry result tells us that there exists a bijective linear transformation ψ sending the sequence $x_1, \dots, x_{n_1}, y$ of $n_1 + 1$ points in general position to the sequence $x_1, \dots, x_{n_1}, \tilde{y}$ of $n_1 + 1$ points in general position. Now note that this means that ψ stabilises P_1 while not stabilising P_2 and so up to scalar multiple we have exhibited an element in $\mathrm{Stab}_{\mathrm{SL_{n+1}}(\mathbb{F})}(P_1)$ that is not in $\mathrm{Stab}_{\mathrm{SL_{n+1}}(\mathbb{F})}(P_2)$, and hence completing the proof. $\square$

Lie algebras

Let us now take $\mathbb{F}$ to be a field with characteristic 0 and let $\mathfrak{sl}_{n+1}(\mathbb{F})$ denote the *special linear Lie algebra* of order $n+1$ of $(n+1)\times(n+1)$ matrices with trace zero and with Lie bracket $[X,Y] := XY - YX$.

Let $\mathfrak{stab}_{n+1}(P)$ be the associated Lie algebra to the Lie group $\mathrm{Stab}_{\mathrm{SL_{n+1}(\mathbb{F})}}(P)$.

We define the incidence system $\mathcal{I}_{(\mathfrak{sl}_{n+1},\mathbb{F})}$. As a set, let

$$\mathcal{I}_{(\mathfrak{sl}_{n+1},\mathbb{F})} := \{\mathfrak{stab}_{n+1}(P), P \in \mathcal{I}_{\mathbb{P}^n}\}.$$

Now $\mathrm{SL}_{n+1}(\mathbb{F})$ acts on the elements of $\mathcal{I}_{(\mathfrak{sl}_{n+1},\mathbb{F})}$ as follows: $g.\mathfrak{stab}_{n+1}(P) := \mathfrak{stab}_{n+1}(g.P)$ for all $g \in \mathrm{SL}_{n+1}(\mathbb{F})$ and so we can identify the set of adjoint orbits of the elements in $\mathcal{I}_{(\mathfrak{sl}_{n+1},\mathbb{F})}$ with the set S of the adjoint orbits of the elements of $\mathcal{I}_{(\mathrm{SL}_{n+1},\mathbb{F})}$. Define the type function $t_{\mathfrak{sl}_{n+1},\mathbb{F}} : \mathcal{I}_{(\mathfrak{sl}_{n+1},\mathbb{F})} \to S$ such that it sends every element to its corresponding orbit. Finally, the incidence relation is defined as follows: $\mathfrak{stab}_{n+1}(P_1), \mathfrak{stab}_{n+1}(P_2) \in \mathcal{I}_{(\mathfrak{sl}_{n+1},\mathbb{F})}$ are incident if their intersection is the associated Lie algebra of a parabolic Lie subgroup of $\mathrm{SL}_{n+1}(\mathbb{F})$.

Theorem 3.1.4. *The map* $\mathrm{Stab}_{\mathrm{SL_{n+1}(\mathbb{F})}}(P) \mapsto \mathfrak{stab}_{n+1}(P)$ *is an incidence system isomorphism* $\mathcal{I}_{(\mathrm{SL}_{n+1},\mathbb{F})} \to \mathcal{I}_{(\mathfrak{sl}_{n+1},\mathbb{F})}$.

Proof. Call this map $\widetilde{\phi}$. It is a map that takes a subgroup to its subalgebra. It is clearly uniquely defined and well-defined. And by construction it is type preserving. Note also that since $\widetilde{\phi}(G_1 \cap G_2) = \widetilde{\phi}(G_1) \cap \widetilde{\phi}(G_2)$ for all $G_1, G_2 \in \mathcal{I}_{(\mathrm{SL}_{n+1},\mathbb{F})}$, $\widetilde{\phi}$ is incidence preserving as well. It is also surjective by definition.

Finally $\widetilde{\phi}$ is injective: Chevalley, see for example [9, page 154], proved that parabolic subgroups of algebraic groups are connected, and we know that over a field of characteristic zero, see for example [26, page 87], there is a one to one correspondence between the connected subgroups of $\mathrm{SL}_{n+1}(\mathbb{F})$ and their Lie algebras, regarded as subalgebras of $\mathfrak{sl}_{n+1}(\mathbb{F})$.

□

The second and third definitions (for Lie groups and Lie algebras) admit obvious generalisations to give buildings associated to any semisimple algebraic group (for this, see [37]) or any semisimple Lie algebra (see [16]). These are called *parabolic buildings*.

3.2 Affine buildings and parahoric buildings

Definition 3.2.1 (Irreducible affine building)**.** An *irreducible affine building* is a building Δ of type Π, where Π is an irreducible affine Coxeter diagram (see Figure 2-3).

Example 3.2.2 (Thin building of type $\widetilde{A_1}$). The thin building of type $\widetilde{A_1}$ is illustrated in Figure 3-2. Its associated incidence system is hence a thin tree (i.e. a graph where every two vertices are connected by exactly one path). See Figure 3-3.

Figure 3-2: *An illustration of the thin affine building of type* $\widetilde{A_1}$

Figure 3-3: *The associated incidence system of the thin affine building of type* $\widetilde{A_1}$

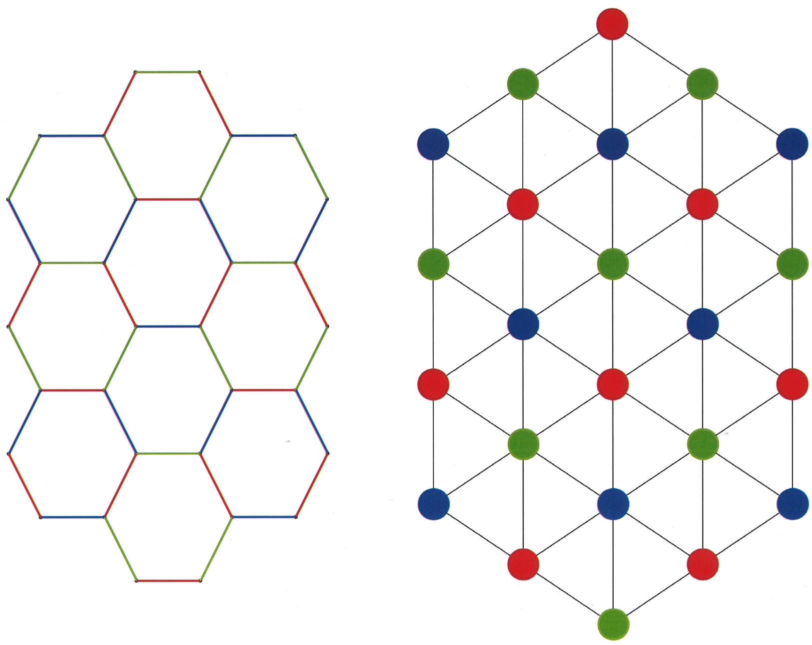

Figure 3-4: *An illustration of the thin affine building of type* $\widetilde{A_2}$ *(on the left) and its associated incidence system (on the right)*

Similarly with spherical buildings there is an important class of affine buildings, namely the so-called 'parahoric buildings'. We look at an example associated to equivalence classes of lattices in $\mathbb{K}^{n+1}$ and one associated to the stabilisers of these lattices.

Equivalence classes of lattices in $\mathbb{K}^{n+1}$

We begin with the definition of a discrete valuation.

Definition 3.2.3 (Discrete valuation)**.** Let $\mathbb{K}$ be a field with multiplicative group of nonzero elements $\mathbb{K}^*$. We call a surjective homomorphism $v : \mathbb{K}^* \to \mathbb{Z}$ a *discrete valuation* if $v(x+y) \geqslant \min\{v(x), v(y)\}$ for all $x, y \in \mathbb{K}^*$ with $x + y \neq 0$.

We extend this map to the whole of $\mathbb{K}$ by sending 0 to ∞.

Note that $v(1) = v(1\cdot 1) = v(1)+v(1) = 0$ and $v(-1)+v(-1) = v(-1\cdot -1) = v(1) = 0$ so $v(-1) = 0$. We also have $v(-x) = v(-1 \cdot x) = v(-1) + v(x) = v(x)$. Therefore the subset $A := \{x \in \mathbb{K} | v(x) \geqslant 0\}$ is a subring of $\mathbb{K}$ and we call it the *valuation ring* associated to $\mathbb{K}$.

Now let $\pi \in \mathbb{K}$ such that $v(\pi) = 1$. The principal ideal πA generated by π is therefore $\{x \in \mathbb{K} | v(x) > 0\}$. Let $x \in A \setminus \pi A$ then $v(x) = 0$ which means that $x \in A^*$ since $x \neq 0$ and $v(x^{-1}) = -v(x) = 0 \geqslant 0$. Therefore πA is a maximal ideal and hence the quotient $\Bbbk := A/\pi A$ is a field. We call $\Bbbk$ the *residue field* associated to the valuation v.

Example 3.2.4 (p-adic valuation and p-adic numbers)*.* We define the *p-adic valuation* val_p as follows. For all $\frac{a}{b} \in \mathbb{Q}^*$, $\mathrm{val}_p(\frac{a}{b}) := \max\{r : p^r | a\} - \max\{r : p^r | b\}$, and extend this to 0: $\mathrm{val}_p(0) = \infty$. The map val_p is a discrete valuation ([6, Proposition 2.4, page 16]). Note that $\mathrm{val}_p(p) = \max\{r : p^r | p\} - \max\{r : p^r | 1\} = 1$ and so the residue field associated to val_p is $\Bbbk := A/pA \cong \mathbb{F}_p$.

Now we define the p-adic norm $|\cdot| : \mathbb{Q} \mapsto \mathbb{Z}$ as follows. For all $x \in \mathbb{Q}$, $|x|_p := p^{-\mathrm{val}_p(x)}$ (where we have by convention $p^{-\infty} := 0$). Finally, the ring of *p-adic numbers* is the completion $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$.

Let $\mathbb{K}$ be with a field with a valuation and let its valuation ring be A as above.

Definition 3.2.5 (Lattice)**.** An *A-lattice* in $\mathbb{K}^{n+1}$ is an A-submodule $L \leqslant \mathbb{K}^{n+1}$ of the form $L = Af_1 \oplus Af_2 \oplus \ldots \oplus Af_{n+1}$ for some basis $f_1, f_2, \ldots, f_{n+1}$ of $\mathbb{K}^{n+1}$.

If $e_1, \ldots, e_{n+1}$ is the standard basis of $\mathbb{K}^{n+1}$ then we call the lattice $L = Ae_1 \oplus \ldots \oplus Ae_{n+1}$ the *standard A-lattice*.

We say that the lattices L and L' are equivalent if there exists $\lambda \in \mathbb{K}$ such that $L = \lambda L'$. Denote the equivalence class of the lattice L by $[L]$ and if $L = Af_1 \oplus Af_2 \oplus \ldots \oplus Af_{n+1}$ where $f_1, f_2, \ldots, f_{n+1}$ is a basis of $\mathbb{K}^{n+1}$, then we denote its equivalence class by $[\![f_1, f_2, \ldots, f_{n+1}]\!]$.

If $L := Af_1 \oplus \ldots \oplus Af_{n+1}$ then define $g.L := A(g.f_1) \oplus \ldots \oplus A(g.f_{n+1})$ where $g \in \mathrm{SL}_{n+1}(\mathbb{K})$, and $g.[L] := [g.L]$.

Let us now define the incidence system $\mathcal{I}_{(\mathbb{K}^{n+1},A)}$. As a set:

$$\mathcal{I}_{(\mathbb{K}^{n+1},A)} := \left\{\text{Equivalence classes of } A-\text{lattices in } \mathbb{K}^{n+1}\right\}.$$

Now let Λ and Λ' be two distinct vertices (equivalence classes of lattices). We say that they are incident if there exist lattices $\lambda \in \Lambda$ and $\lambda' \in \Lambda'$ such that

$$\pi\lambda \lneqq \lambda' \lneqq \lambda.$$

Finally let S be the set of orbits of these equivalence classes of lattices and let the type function be $t_{(\mathbb{K}^{n+1},A)} : \mathcal{I}_{(\mathbb{K}^{n+1},A)} \to S$ sending every equivalence class to its corresponding orbit.

Theorem 3.2.6 ([1]). *The associated building of the incidence system $\mathcal{I}_{(\mathbb{K}^{n+1},A)}$ is an affine building of type $\widetilde{A_n}$.*

The group SL_{n+1} over the field $\mathbb{K}$ with valuation v

For what follows, let $\mathbb{K}$ be a field with a valuation v, a valuation ring A and a residue field $\mathbb{k}$ as defined above.

Let Λ be an equivalence class of lattices in $\mathbb{K}^{n+1}$ and let us define

$$\mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}(\Lambda) := \{g \in \mathrm{SL}_{n+1}(\mathbb{K}) | g.\Lambda = \Lambda\}.$$

We now define the incidence system $\mathcal{I}_{(\mathrm{SL}_{n+1},\mathbb{K},v)}$. As a set

$$\mathcal{I}_{(\mathrm{SL}_{n+1},\mathbb{K},v)} := \left\{\mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}(\Lambda) | \Lambda \in \mathcal{I}_{(\mathbb{K}^{n+1},A)}\right\}.$$

The types of $\mathcal{I}_{(\mathrm{SL}_{n+1},\mathbb{K},v)}$ are the adjoint orbits when letting $\mathrm{SL}_{n+1}(\mathbb{K})$ act on $\mathcal{I}_{(\mathrm{SL}_{n+1},\mathbb{K},v)}$. But note that $\Lambda_2 = g.\Lambda_1$ implies that $\mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}(\Lambda_2) = g.\,\mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}(\Lambda_1)$. Therefore we can identify the types of $\mathcal{I}_{(\mathrm{SL}_{n+1},\mathbb{K},v)}$ with the types of $\mathcal{I}_{(\mathbb{K}^{n+1},A)}$.

Finally we need to define incidence in $\mathcal{I}_{(\mathrm{SL}_{n+1},\mathbb{K},v)}$. For this, we first define the so-called 'parahoric' subgroups. This is the analogous concept to parabolic subgroups previously defined. We say that a subgroup of $\mathrm{SL}_{n+1}(\mathbb{K})$ is *parahoric* if it contains the stabiliser of a full flag in $\mathcal{I}_{(\mathbb{K}^{n+1},A)}$. Then $\mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}(\Lambda_1)$ and $\mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}(\Lambda_2)$ are said to be incident if their intersection is parahoric.

Theorem 3.2.7. *The map $\Lambda \mapsto \mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}(\Lambda)$ is an incidence system isomorphism $\mathcal{I}_{(\mathbb{K}^{n+1},A)} \to \mathcal{I}_{(\mathrm{SL}_{n+1},\mathbb{K},v)}$.*

Proof. Call this map ψ. It is clearly well-defined, type and incident preserving.

The map ψ is by construction a surjective map, and so we only need to show that it is injective. Suppose that $\mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}([\lambda_1]) = \mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}([\lambda_2])$. We want to show that $[\lambda_1] = [\lambda_2]$.

We know that $\mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}([\lambda_1]) = \mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}([\lambda_2])$ implies that $\mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}(\lambda_1) = \mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}(\lambda_2)$. Now since the group $\mathrm{GL}_{n+1}(\mathbb{K})$ acts on the lattices freely and transitively so $\lambda_1 = h\lambda_2$ for some $h \in \mathrm{GL}_{n+1}(\mathbb{K})$. And so if $g \in \mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}([\lambda_1])$ then $g.\lambda_1 = \lambda_1$ and $g.\lambda_2 = \lambda_2$. Thus, $g.(h\lambda_2) = h\lambda_2$ and so $(h^{-1}gh).\lambda_2 = \lambda_2$ implying that $h^{-1}gh \in \mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}(\lambda_2) = \mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}(\lambda_1)$. Thus

$$\mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}(\lambda_1) \subseteq h\,\mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}(\lambda_1)h^{-1}. \tag{3.2.1}$$

Now since $g.\lambda_2 = \lambda_2$ we have that $g.(h^{-1}\lambda_1) = h^{-1}\lambda_1$ and so $(hgh^{-1}).\lambda_1 = \lambda_1$, implying that $hgh^{-1} \in \mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}(\lambda_1)$, therefore

$$h\,\mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}(\lambda_1)h^{-1} \subseteq \mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}(\lambda_1). \tag{3.2.2}$$

Equations (3.2.1) and (3.2.2) imply that $\mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}(\lambda_1) = h\,\mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}(\lambda_1)h^{-1}$, thus h is in the normaliser, in $\mathrm{GL}_{n+1}(\mathbb{K})$, of $\mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}(\lambda_1)$. This in turn implies that $h = kh'$, for some $k \in \mathbb{K}^*$ and h' is in the normaliser, in $\mathrm{SL}_{n+1}(\mathbb{K})$, of $\mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}(\lambda_1) = \mathrm{Stab}_{\mathrm{SL}_{n+1}(\mathbb{K})}(\lambda_2)$. Finally $\lambda_1 = h\lambda_2$ implies that $\lambda_1 = kh'\lambda_2 = k\lambda_2$ which in turn implies that $[\lambda_1] = [\lambda_2]$.

□

The buildings associated to $\mathcal{I}_{(\mathbb{K}^{n+1},A)}$ and $\mathcal{I}_{(\mathrm{SL}_{n+1},\mathbb{K},v)}$ are known as *parahoric buildings.*

Finally, we would like to know how 'thick' this incidence system is. As it turns out the number of incident vertices to any given one is linked to the cardinality of the residue field $\Bbbk$ as shown in the following proposition.

Proposition 3.2.8 ([1])**.** *Let $x = \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{K})}(\Lambda)$ be any vertex in $\mathcal{I}_{(\mathrm{SL}_2,\mathbb{K},v)}$. Then the set of vertices incident to x can be identified with a projective line over the residue field $\Bbbk$.*

For example, in the case where $\mathbb{K} = \mathbb{Q}_2$ (the field of 2-adic numbers) with the 2-adic valuation v_2 (as defined earlier), π can be taken to be the integer 2 and so $\Bbbk = A/2A \cong \mathbb{F}_2 = \{0, 1\}$. Then

$$\mathcal{I}_{(\mathrm{SL}_2,\mathbb{Q}_2,v_2)} = \left\{\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Q}_2)}(\Lambda)\,|\Lambda \text{ is an equivalence class of } A-\text{lattices in } \mathbb{Q}_2^2\right\}.$$

Now the previous proposition implies that for every vertex in $\mathcal{I}_{(\mathrm{SL}_2,\mathbb{Q}_2,v_2)}$ the set of incident vertices to it can be identified with the projective line over $\mathbb{F}_2$ which has 3 points. This is illustrated in Figure 3-5 below.

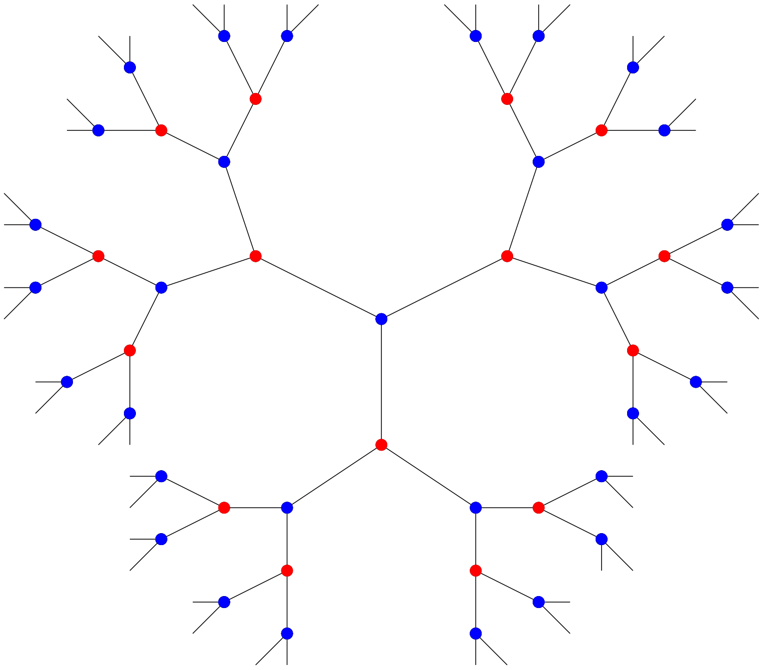

Figure 3-5: *The incidence system* $\mathcal{I}_{(\mathrm{SL}_2,\mathbb{Q}_2,v_2)}$ *associated to the affine building of type* $\widetilde{A_1}$

CHAPTER 4

TWIN BUILDINGS

4.1 Definitions

Twin buildings were first introduced by Tits in [41]. Let $(\mathcal{B}_+, \delta_+)$ and $(\mathcal{B}_-, \delta_-)$ be two buildings of the same type Π with Coxeter system (W, I). Let $\delta : (\mathcal{B}_+ \times \mathcal{B}_-) \cup (\mathcal{B}_- \times \mathcal{B}_+) \to W$ be a map. Tits requires that these data $(\mathcal{B}_+, \mathcal{B}_-, \delta)$ satisfy that for each $\epsilon \in \{+, -\}$, any chamber $a \in \mathcal{B}_\epsilon$, and any $b \in \mathcal{B}_{-\epsilon}$, where $w := \delta(a, b)$:

(1) $\delta(a, b) = \delta(b, a)^{-1}$,

(2) If $a' \in \mathcal{B}_\epsilon$ satisfies $\delta_\epsilon(a', a) = i$ with $i \in I$ and the length of iw is strictly smaller than the length of w, then $\delta(a', b) = iw$.

(3) For any $i \in I$, there exists a chamber $a' \in \mathcal{B}_\epsilon$ with $\delta_\epsilon(a, a') = i$ and $\delta_\epsilon(a', b) = iw$.

These axioms are straightforwardly equivalent to the following:

Definition 4.1.1 (Twin building and codistance)**.** Let $\mathcal{B}_+$ and $\mathcal{B}_-$ be as above and let $\delta : \mathcal{B}_+ \times \mathcal{B}_- \to W$ be a map such that for any i-adjacent chambers a and a' in $\mathcal{B}_+$ then $\delta(a', b) \in \{\delta(a, b), i\delta(a, b)\}$ for all $b \in \mathcal{B}_-$, and similarly for any i-adjacent chambers b and b' in $\mathcal{B}_-$ then $\delta(a, b') \in \{\delta(a, b), \delta(a, b)i\}$ for all $a \in \mathcal{B}_+$. We say that the triple $(\mathcal{B}_+, \mathcal{B}_-, \delta)$ is a *twin building* of type Π and we call the map δ a *codistance* if the following condition is satisfied:

For any chamber $b \in \mathcal{B}_-$, there is a unique chamber a in every i-panel in $\mathcal{B}_+$ such that the length of $\delta(a, b)$ is strictly greater than $\delta(a', b)$ for any other chamber a' in this i-panel. Similarly, for any $a \in \mathcal{B}_+$ there exists a unique chamber b in every i-panel in $\mathcal{B}_-$ such that the length of $\delta(a, b)$ is strictly greater than $\delta(a, b')$ for any other chamber b' in this i-panel.

Now we would like to give an alternative but equivalent definition using the so-called opposite apartments.

Let $\mathcal{B}$ be a chamber system. Let $\delta : \mathcal{B} \times \mathcal{B} \to W$ be a map such that if $\delta(a, b) = \delta(a, b)^{-1}$ for all chambers $a, b \in \mathcal{B}$ and for any i-adjacent chambers a and a' then $\delta(a', b) \in \{\delta(a, b), i\delta(a, b)\}$ for all $b \in \mathcal{B}$. Suppose that $\mathcal{B}$ is the disjoint union, in the category of chamber systems, of two connected chamber systems $\mathcal{B}_+$ and $\mathcal{B}_-$ (i.e. the vertex set of $\mathcal{B}$ is the disjoint union of the vertex set of $\mathcal{B}_+$ and the vertex set of $\mathcal{B}_-$, while the labelled edges of $\mathcal{B}$ are the labelled edges in $\mathcal{B}_+$ and $\mathcal{B}_-$ only). If $a_1 \in \mathcal{B}_+$ and $a_2 \in \mathcal{B}_-$ are two chambers such that $\delta(a_1, a_2) = 1_W$, then we say that they are *δ-opposite chambers*. Now let A_+ be an apartment in $\mathcal{B}_+$ and let A_- be an apartment in $\mathcal{B}_-$ such that for all chambers $a_1 \in A_+$ there exists a δ-opposite chamber a_2 in A_- and for all chambers a_2 in A_- there exists a δ-opposite chamber a_1 in A_+, then we say that A_+ and A_- are *δ-opposite apartments*.

Definition 4.1.2 (Pre-twin building)**.** Let $\mathcal{B} = \mathcal{B}_+ \sqcup \mathcal{B}_-$ and $\delta : \mathcal{B} \times \mathcal{B} \to W$ be as above. We say that $\mathcal{B}$ together with such a map δ is a *pre-twin building* if the following conditions are satisfied.

(1) For all chambers a, b in $\mathcal{B}_+$ there exists a $\delta_{|\mathcal{B}_+ \times \mathcal{B}_+}$-compatible apartment A in $\mathcal{B}_+$ such that $a, b \in A$. Similarly, for all chambers a, b in $\mathcal{B}_-$ there exists a $\delta_{|\mathcal{B}_- \times \mathcal{B}_-}$-compatible apartment A in $\mathcal{B}_-$ such that $a, b \in A$.

(2) For all $a \in \mathcal{B}_+$ and $b \in \mathcal{B}_-$, there exist two δ-opposite apartments A_+ in $\mathcal{B}_+$ and A_- in $\mathcal{B}_-$ such that $a \in A_+$ and $b \in A_-$.

For the next theorem, we will need the following lemma.

Lemma 4.1.3. *[1, Proposition 5.179 (3), page 279] Let $(\mathcal{B}_+, \mathcal{B}_-, \delta)$ be a twin building, and let $a \in \mathcal{B}_+$ and $b \in \mathcal{B}_-$. Then there exist two δ-opposite apartments $A_+ \in \mathcal{B}_+$ and $A_- \in \mathcal{B}_-$ such that $a \in A_+$ and $b \in A_-$.*

Theorem 4.1.4. *Any twin building is a pre-twin building.*

Proof. Let $(\mathcal{B}_+, \mathcal{B}_-, \delta)$ be a twin building in the sense of Definition 4.1.1. Let

$\mathcal{B} := \mathcal{B}_+ \sqcup \mathcal{B}_-$ and define the map

$$\begin{aligned} \delta^* : \mathcal{B} \times \mathcal{B} &\to W \\ (a,b) &\mapsto \begin{cases} \delta_+(a,b), \text{ if } a,b \in \mathcal{B}_+, \\ \delta_-(a,b), \text{ if } a,b \in \mathcal{B}_-, \\ \delta(a,b), \text{ if } a \in \mathcal{B}_+ \text{ and } b \in \mathcal{B}_-, \\ \delta(b,a)^{-1}, \text{ if } a \in \mathcal{B}_- \text{ and } b \in \mathcal{B}_+. \end{cases} \end{aligned}$$

Let us now show that $(\mathcal{B}, \delta^*)$ is a pre-twin building in the sense of Definition 4.1.2. Let $a, b \in \mathcal{B}$. If both chambers a and b belong to either $\mathcal{B}_+$ or $\mathcal{B}_-$ then $\delta^*(a,b) = \delta_+(a,b) = \delta_+(b,a)^{-1} = \delta^*(b,a)^{-1}$ or $\delta^*(a,b) = \delta_-(a,b) = \delta_-(b,a)^{-1} = \delta^*(b,a)^{-1}$ since $\mathcal{B}_+$ and $\mathcal{B}_-$ are buildings.

(1) Let $a, b \in \mathcal{B}_+$. Then there exists a δ_+-compatible (i.e. $\delta^*_{|\mathcal{B}_+ \times \mathcal{B}_+}$-compatible) apartment in $\mathcal{B}_+$ containing a and b since $\mathcal{B}_+$ is a building. A similar argument holds in the case $a, b \in \mathcal{B}_-$ instead.

(2) We want to show that if $a \in \mathcal{B}_+$ and $b \in \mathcal{B}_-$, there exist two δ-opposite apartments $A_+ \in \mathcal{B}_+$ and $A_- \in \mathcal{B}_-$ such that $a \in A_+$ and $b \in A_-$. This follows directly from Lemma 4.1.3. $\square$

The other direction, namely that pre-twin buildings are twin buildings, was proved by Abramenko and Ronan in their paper *A Characterization of Twin Buildings by Twin Apartments* (see [2]).

4.2 Twin trees (incidence systems)

Ronan and Tits investigated in their paper [35] in detail the 1-dimensional affine case of twin buildings. These are called twin trees. The approach they take is that of incidence systems. We present in this section their definition of twin trees together with a useful example of such an object.

4.2.1 Definitions

Let T be a set and call its elements vertices. Let $\mathrm{dist} : T \times T \to \mathbb{N}$ be a map such that $\mathrm{dist}(x,y) = \mathrm{dist}(y,x)$ for all vertices $x, y \in T$ and such that $\mathrm{dist}(x,y) = 0$ if and only if $x = y$. We call this map the *distance* between x and y. When $\mathrm{dist}(x,y) = 1$ we say that x and y are *adjacent*.

Definition 4.2.1 (Tree)**.** We call the system (T, dist) a *tree* when the following condition is satisfied: If $\mathrm{dist}(x, y) = m$ then for all y' adjacent to y, $\mathrm{dist}(x, y') = m \pm 1$, and if $m > 0$ there is a unique such y' with $\mathrm{dist}(x, y') = m - 1$

By joining every adjacent pair by an edge, we obtain a tree in the usual sense of a connected graph with no circuits. From now on we will consider only trees with every vertex having at least two other vertices adjacent to them.

Now let T_+ and T_- be a pair of trees, then a map $\mathrm{codist} : (T_+ \times T_-) \cup (T_- \times T_+) \to \mathbb{N}$ is called a *codistance* if $\mathrm{codist}(x, y) = \mathrm{codist}(y, x)$ for all vertices $(x, y) \in (T_+ \times T_-) \cup (T_- \times T_+)$.

Definition 4.2.2 (Twin tree)**.** We call the system $(T_+, T_-, \mathrm{codist})$ a *twin tree* when the following condition is satisfied: If $\mathrm{codist}(x, y) = m$ then for all y' adjacent to y, $\mathrm{codist}(x, y') = m \pm 1$, and if $m > 0$ there is a unique such y' with $\mathrm{codist}(x, y') = m + 1$.

The following lemma is straightforward from the definitions.

Lemma 4.2.3. *Let* $(T_+, T_-, \mathrm{codist})$ *be a twin tree. Then there are* $x \in T_+$ *and* $y \in T_-$ *such that* $\mathrm{codist}(x, y) = 0$.

4.2.2 The field $\mathbb{K} = \mathbb{k}(t)$ of rational functions over a field $\mathbb{k}$, with valuations v_0 and v_∞

Let $\mathbb{k}$ be a field. Let $\mathbb{K} = \mathbb{k}(t) := \left\{ \frac{p(t)}{q(t)} | p, q \in \mathbb{k}[t] \right\}$ be the field of rational functions over the field $\mathbb{k}$ with the projective line $\mathbb{P}(\mathbb{k}^2)$ as their domain.

Now define the valuation v_0 as follows

$$\begin{aligned} v_0 \colon \mathbb{K} &\to \mathbb{Z} \cup \{\infty\} \\ f &\mapsto m - n, \end{aligned}$$

where $f(t) = \frac{p(t)}{q(t)}$, $p(t) = \lambda_m t^m + \lambda_{m-1} t^{m+1} + \ldots + \lambda_M t^M$ and $q(t) = \mu_n t^n + \mu_{n+1} t^{n+1} + \ldots + \mu_N t^N$ for some $m, M, n, N \geqslant 0$ and such that $\lambda_m \neq 0$, $\lambda_M \neq 0$, $\mu_m \neq 0$ and $\mu_M \neq 0$. The polynomials p and q are taken to be coprime.

The valuation ring is

$$A_0 := \{x \in \mathbb{K} | v_0(x) \geqslant 0\} = \left\{ \frac{p(t)}{q(t)} | p, q \in \mathbb{k}[t] \text{ and 0 is not a root of } q \right\},$$

and π can be taken to be $f(t) = t$ ($v_0(t) = 1$). So the residue field is $A_0/tA_0 \cong \mathbb{k}$.

We also define the valuation v_∞ as follows

$$\begin{aligned} v_\infty \colon \mathbb{K} &\to \mathbb{Z} \cup \{\infty\} \\ f &\mapsto N - M. \end{aligned}$$

Where the associated valuation ring is

$$A_\infty := \{x \in \mathbb{K} | v_\infty(x) \geqslant 0\} = \left\{ \frac{p(t)}{q(t)} | p, q \in \Bbbk[t] \text{ as above with } N \geqslant M \right\},$$

and we let $\pi := \frac{1}{t}$ $(v_\infty(t^{-1}) = 1)$. So the residue field $A_\infty / t^{-1} A_\infty \cong \Bbbk$.

4.2.3 Example

Let $\mathcal{I}_{(\Bbbk(t)^2, A_0)}$ and $\mathcal{I}_{(\Bbbk(t)^2, A_\infty)}$ be the associated incidence systems to the parahoric buildings of SL_2 over $\Bbbk(t)$ with valuation v_0 and valuation v_∞ respectively (which are buildings of type $\widetilde{A_1}$ as we saw in the previous chapter). The residue field in both cases is $\Bbbk$ and so if we pick $\Bbbk := \mathbb{F}_2$ then $|\Bbbk| = 2$, and so the trees $\mathcal{I}_{(\Bbbk(t)^2, A_0)}$ and $\mathcal{I}_{(\Bbbk(t)^2, A_\infty)}$ are as illustrated in Figure 3-5.

For simplicity and for the remainder of this thesis, let us denote by $\mathcal{T}_0$ the tree $\mathcal{I}_{(\Bbbk(t)^2, A_0)}$ and by $\mathcal{T}_\infty$ the tree $\mathcal{I}_{(\Bbbk(t)^2, A_\infty)}$.

Lemma 4.2.4. *Let $x \in \mathcal{T}_0$ and $y \in \mathcal{T}_\infty$. Then we can choose λ_x a $\Bbbk[t]$-lattice in $\Bbbk(t)^2$ and λ_y a $\Bbbk[t^{-1}]$-lattice in $\Bbbk(t)^2$ such that $x = [\lambda_x]$ and $y = [\lambda_y]$ and such that $\lambda_x \cap \lambda_y$ contains a basis of $\Bbbk(t)^2$ but $t\lambda_x \cap \lambda_y$ does not contain such a basis.*

Proof. Suppose $\lambda_x \cap \lambda_y$ does contain a basis of $\Bbbk(t)^2$ but so does $t\lambda_x \cap \lambda_y$. We want to find another lattice λ'_x such that $x = [\lambda'_x]$ and such that $\lambda'_x \cap \lambda_y$ contains a basis of $\Bbbk(t)^2$ but $t\lambda'_x \cap \lambda_y$ does not. It is clear that for n great enough $t^n \lambda_x \cap \lambda_y$ is the empty set. And so for some $0 < m < n$, $t^m \lambda_x \cap \lambda_y$ contains a basis while $t^{m+1}\lambda_x \cap \lambda_y$ does not. Finally we can pick λ'_x to be $t^m \lambda_x$.

Otherwise suppose that $\lambda_x \cap \lambda_y$ does not contain a basis for $\Bbbk(t)^2$. Let $\lambda_x = [f_1, f_2]$ and $\lambda_y = [g_1, g_2]$, for some bases f_1, f_2 and g_1, g_2. So $g_1 = \alpha_1(t) f_1 + \beta_1(t) f_2$ for some $\alpha_1(t), \beta_1(t) \in \Bbbk(t)$ and similarly $g_2 = \alpha_2(t) f_1 + \beta_2(t) f_2$ for some $\alpha_2(t), \beta_2(t) \in \Bbbk(t)$. Therefore, for n large enough, we have: $g_1 = (t^n \alpha_1(t))\, t^{-n} f_1 + (t^n \beta_1(t))\, t^{-n} f_2$ and $g_2 = (t^n \alpha_2(t))\, t^{-n} f_1 + (t^n \beta_2(t))\, t^{-n} f_2$, where all $t^n \alpha_1(t)$, $t^n \beta_1(t)$, $t^n \alpha_2(t)$ and $t^n \beta_2(t)$ are all in $\Bbbk[t]$ and so pick $\lambda'_x = t^{-n} \lambda_x$. $\square$

Then let us define the map d such that $d(x, y) = d(y, x)$ for all $(x, y) \in$

$(\mathcal{T}_0 \times \mathcal{T}_\infty) \cup (\mathcal{T}_\infty \times \mathcal{T}_0)$ as follows:

$$\begin{aligned} d\colon (\mathcal{T}_0 \times \mathcal{T}_\infty) \cup (\mathcal{T}_\infty \times \mathcal{T}_0) &\to \mathbb{N} \\ (x, y) &\mapsto \dim_{\Bbbk}(t\lambda_x \cap \lambda_y). \end{aligned}$$

This is a well-defined map and is indeed a codistance [35, Proposition 2.2, page 468], showing that $(\mathcal{T}_0, \mathcal{T}_\infty, d)$ is a twin tree.

4.3 Twin trees (buildings)

We explain in this section how one goes from a twin tree (in the context of incidence systems, as defined above) to a twin tree (in the context of buildings, as we will see below), and vice versa.

Let T_+, T_- be a pair of incidence systems over $I = \{\circ, \bullet\}$ and $I' = \{\color{red}\bullet, \color{blue}\bullet\}$ and let (T_+, T_-, d) be a twin tree, as defined above in Section 4.2. We would like to identify the sets I and I' together but not in a random way. We do this in the following way: from Lemma 4.2.3 we know that there exist $x \in T_+$ and $y \in T_-$ such that $d(x, y) = 0$. Say, without loss of generality, that the type of x is $\circ$ and the type of y is $\color{red}\bullet$, then we identify $\circ$ with $\color{red}\bullet$ and hence $\bullet$ with $\color{blue}\bullet$. This choice is needed in proving the two propositions below.

Remark 4.3.1. Since $d(x, y) = 0$ (where the type of x is $\circ$ and the type of y is $\color{red}\bullet$) it is straightforward from the definition of the codistance that for $x' \in T_+$ and $y' \in T_-$, $d(x', y')$ is even if and only if either the type of x' is $\circ$ and the type of y' is $\color{red}\bullet$ or the type of x'' is $\bullet$ and the type of y' is $\color{blue}\bullet$.

Now let W be the Weyl group generated by $I = \{\circ, \bullet\}$ as in previous chapters, and let $\mathcal{B}_+$ be the building of T_+ and let $\mathcal{B}_-$ be the one of T_-. We want to define a codistance $\delta : (\mathcal{B}_+ \times \mathcal{B}_-) \cup (\mathcal{B}_- \times \mathcal{B}_+) \to W$.

Let $a \in \mathcal{B}_+$ and $b \in \mathcal{B}_-$, $\exists x_a, x'_a \in T_+$ and $y_b, y'_b \in T_-$ such that x_a—x'_a in T_+ is associated to a and y_b—y'_b in T_- is associated to b. Let

$$S := (d(x_a, y_b), d(x_a, y'_b), d(y_b, x'_a), d(x'_a, y'_b))$$

where, without loss of generality and up to reordering, $d(x_a, y_b) \geqslant s$ for all $s \in S$.

Lemma 4.3.2. *Either* $S = (1, 0, 0, 1)$ *or* $S = (n + 1, n, n, n - 1)$ *for some* $n \in \mathbb{N}^*$.

Proof. Clearly not all of the elements of S are 0 since by definition if $d(x,y) = 0$ then $d(x,y') = 1$ for all y' adjacent to y. And so $d(x_a, y_b) \neq 0$. Thus let $d(x_a, y_b) = n+1$ for some $n \in \mathbb{N}^*$ and so by the choice of $d(x_a, y_b)$ maximal, $d(x_a, y'_b) = n$ and $d(y_b, x'_a) = n$ since x'_a is adjacent to x_a and y'_b is adjacent to y_b. Now there are two possible cases: $n = 0$ and so $d(x'_a, y'_b) = 1$ and $S = (1,0,0,1)$. Otherwise, $n \neq 0$ and so $d(x'_a, y'_b) = n \pm 1$. Suppose it is $n+1$. Now since $d(x_a, y_b) = n+1$ and $d(x_a, y'_b) = n$, we know that there exists another vertex y''_b adjacent to y_b such that $d(x_a, y''_b) = n+2$. But now since $d(x'_a, y_b) = n$ and $d(x'_a, y'_b) = n+1$ then from the definition, all other adjacent vertices to y_b must be at a codistance of $n-1$ from x'_a, which implies that $d(y''_b, x'_a) = n-1$. This in turns implies that $d(x_a, y''_b)$ is either n or $n-2$, which leads to a contradiction since we previously found that it must be $n+2$. Therefore $d(x'_a, y'_b) = n-1$ and so $S = (n+1, n, n, n-1)$ as desired. □

If $S = (1,0,0,1)$, define $\delta(a,b) := 1_W$. Otherwise, $S = (n+1, n, n, n-1)$ with $n \neq 0$ and we let $\delta(a,b)$ be the alternating word of length n beginning with the type of x_a. Finally, let $\delta(b,a) := \delta(a,b)^{-1}$, for all $b \in \mathcal{B}_-$ and $a \in \mathcal{B}_+$.

Proposition 4.3.3. *The map $\delta : (\mathcal{B}_+ \times \mathcal{B}_-) \cup (\mathcal{B}_- \times \mathcal{B}_+) \to W$ as defined above is a codistance.*

Proof. Let $\epsilon \in \{+,-\}$ and let a be any chamber in $\mathcal{B}_\epsilon$ and b any chamber in $\mathcal{B}_{-\epsilon}$ with $w := \delta(a,b)$.

(1) Then we have $\delta(b,a) := \delta(a,b)^{-1}$ by definition.

Let us assume, for the rest of the proof, that $\epsilon = +$ (the case $\epsilon = -$ is similar). There are incident vertices x_a—x'_a in T_ϵ associated to a and y_b—y'_b in $T_{-\epsilon}$ associated to b. Let $w := \delta(a,b)$ and let S be defined as previously, i.e. $S := (d(x_a, y_b), d(x_a, y'_b), d(y_b, x'_a), d(x'_a, y'_b)) = (n+1, n, n, n-1)$ where n is the length of the word w. So by definition, w starts with the type of x_a.

(2) Suppose that there exists a chamber $a' \in \mathcal{B}_\epsilon$ satisfying $\delta_\epsilon(a', a) = i$ with $i \in I = \{\circ, \bullet\}$ and such that the length of iw is strictly smaller than the length of w. This means that w is a word that starts with i, which implies that i must be the type of x_a. Thus there is a vertex $x''_a \neq x_a$ of type i incident to x'_a where x'_a—x''_a in T_ϵ is associated to a'. Now we either have $n = 1$ or $n > 1$. If $n = 1$ (implying that $w = i$) and so since $d(y_b, x_a) = 2$ and $d(y_b, x'_a) = 1$ then $d(y_b, x''_a) = 0$. We also know that $d(y'_b, x''_a) = 1$ since $d(x'_a, y'_b) = 0$. Thus $S' = (d(x'_a, y'_b), d(x'_a, y_b), d(y'_b, x''_a), d(x''_a, y'_b)) = (1,0,0,1)$ and so $\delta(a', b) =$

$1_W = iw$. Otherwise, if $n > 1$ then $d(y_b, x''_a)$ must be equal to $n-1$, and so $S' := (d(x'_a, y_b), d(x'_a, y'_b), d(x''_a, y_b), d(x''_a, y'_b)) = (n, n-1, n-1, n-2)$. This means $\delta(a', b)$ starts with the type of x'_a and is of length $n-1$, so in other words $\delta(a', b) = i\delta(a, b)$.

(3) Finally, we want to show that for any $i \in I$, there exists a chamber $a' \in \mathcal{B}_+$ with $\delta_+(a, a') = i$ and $\delta_+(a', b) = iw$. For the case when i is the type of x_a then the result follows from what we proved above. So suppose that i is the type of x'_a instead. Now from the definition of d we know that there exists a unique vertex x''_a incident to x_a such that $d(y_b, x''_a)$ is equal to $d(y_b, x_a) + 1 = (n+1) + 1 = n+2$. Let us call a' the chamber in $\mathcal{B}_+$ associated to x_a—x''_a. Thus $\delta(a', b)$ is a word of length $n+1$ starting with i, so in other words $\delta(a', b) = iw$.

□

Let us now look at it the other way around, namely let $(\mathcal{B}_+, \mathcal{B}_-, \delta)$ be a twin building of type $\widetilde{A_1}$. We want to construct a twin tree associated to it.

Let T_+ be the associated incidence system of $\mathcal{B}_+$ and let T_- be the one associated to $\mathcal{B}_-$. We now want to define a codistance $d : (T_+ \times T_-) \cup (T_- \times T_+) \to \mathbb{N}$.

Let $x \in T_+$ and $y \in T_-$. Now pick a vertex x' adjacent to x so that x—x' is associated to a chamber in $\mathcal{B}_+$, call it a. Similarly, we can pick a vertex y' adjacent to y so that y—y' is associated to a chamber b in $\mathcal{B}_-$. Let the type of x be $i \in I = \{\circ, \bullet\}$ and let the type of y be $j \in \{\circ, \bullet\}$. If $\delta(a, b) = 1_W$ then let $d(x, y) := 0$ if $i = j$ and $d(x, y) := 1$ otherwise. Now suppose $\delta(a, b) \neq 1_W$, and let its length be $n \neq 0$. We begin with the case where n is even. Then we define $d(x, y) := n$ if $i = j$. Otherwise if $i \neq j$ then if $\delta(a, b)$ starts with i let $d(x, y) := n + 1$, and if it starts instead with j then define $d(x, y) := n - 1$. Now if n is odd, then $d(x, y) := n$ if $i \neq j$. Otherwise if $\delta(a, b)$ starts with $i (= j)$ then $d(x, y) := n+1$, and if it starts instead with $i' \in I$ with $i' \neq i$ then $d(x, y) := n-1$.

Finally, we define $d(y, x) := d(x, y)$, for all $y \in T_-$ and $x \in T_+$.

Proposition 4.3.4. *The map $d : (T_+ \times T_-) \cup (T_- \times T_+) \to \mathbb{N}$ is well-defined.*

Proof. Let $x \in T_+$ of type $i \in I$ and let $y \in T_-$ of type $j \in I$. There are a total of four different cases depending on the types of x and y: $i = j = \bullet$, $i = j = \circ$, $i = \bullet$ and $j = \circ$ or $i = \circ$ and $j = \bullet$. Let us do the case $i = \bullet$ and $j = \circ$ in detail, then the other cases are done in a similar way.

As in the definition, pick a vertex x' adjacent to x so that x—x' is associated to a chamber in $\mathcal{B}_+$, call it a. Similarly, we can pick a vertex y' adjacent to y so that y—y' is associated to a chamber b in $\mathcal{B}_-$. Let the length of $\delta(a,b)$ be $n \in \mathbb{N}$. We have the following cases:

(1) n is odd and $\delta(a,b)$ as a word in W starts with $\bullet$. So $d(x,y) = n$.

In order to check that d is well-defined we pick a different vertex $x'' \neq x'$ that is adjacent to x so that x—x'' is associated to a new chamber in $\mathcal{B}_+$, call it a', and we see if we find the same result for $d(x,y)$ using $\delta(a',b)$ instead of $\delta(a,b)$. The case where we take an adjacent vertex y'' to y in $\mathcal{B}_-$ is done similarly.

From the definition of δ we know that there exists a chamber a'' in $\mathcal{B}_+$ that is $\circ$ adjacent to a such that $\delta(a'',b) = \circ\delta(a,b)$, which implies that $\delta(a'',b)$ is of length $n+1$ and starts with $\circ$ (since $\delta(a,b)$ has length n and starts with $\bullet$). Now there are two scenarios. Either $a'' = a'$ and so from the definition of d we get that $d(x,y) = (n+1) - 1 = n$. Otherwise, if $a'' \neq a$ then by transitivity of adjacency, a'' is $\circ$-adjacent to a', and so from the definition of δ we have that $\delta(a',b) = \circ\delta(a'',b) = \circ\,(\circ\delta(a,b)) = \delta(a,b)$, and so $d(x,y) = n$ as desired.

(2) n is odd and $\delta(a,b)$ as a word in W starts with $\circ$. So $d(x,y) = n$.

As before we pick another $x'' \neq x'$ adjacent to x, so x—x'' is associated to a new chamber in $\mathcal{B}_+$, call it a'. Now a' is $\circ$-adjacent to a and so the length of $\circ\delta(a,b)$ is strictly smaller than $\delta(a,b)$, thus by definition of δ, we have that $\delta(a',b)$ is of length $n-1$ and starts with $\bullet$ which implies that $d(x,y)$ should be equal to $(n-1)+1 = n$.

(3) n is even and $\delta(a,b)$ starts with $\bullet$. So $d(x,y) = n+1$.

Pick a vertex $x'' \neq x'$ adjacent to x, so x—x'' is associated to a new chamber in $\mathcal{B}_+$, call it a'. Now from the definition of δ we know that there exists a chamber a'' in $\mathcal{B}_+$ that is $\circ$ adjacent to a such that $\delta(a'',b) = \circ\delta(a,b)$, which implies that $\delta(a'',b)$ is of length $n+1$ and starts with $\circ$ (since $\delta(a,b)$ has length n and starts with $\bullet$). Now there are two cases. Either $a'' = a'$ and so from the definition of d we get that $d(x,y) = n+1$. Otherwise, if $a'' \neq a$ then by transitivity of adjacency, a'' is $\circ$-adjacent to a', and so from the definition of δ we have that $\delta(a',b) = \circ\delta(a'',b) = \circ\,(\circ\delta(a,b)) = \delta(a,b)$, and so $d(x,y) = n+1$ as desired.

(4) The last case now: n is even and $\delta(a,b)$ starts with $\circ$. So $d(x,y) = n-1$.

Pick another $x'' \neq x'$ adjacent to x, so x—x'' is associated to a new chamber in $\mathcal{B}_+$, call it a'. Now a' is $\circ$-adjacent to a and so the length of $\circ\delta(a,b)$ is strictly smaller than $\delta(a,b)$, thus by definition of δ, we have that $\delta(a',b)$ is of length $n-1$ and starts with $\bullet$ which implies that $d(x,y)$ should be equal to $n-1$.

This finishes the proof for the case where x is of type $\bullet$ and y of type $\circ$. The three other cases (namely: the type of both x and y is $\bullet$ or $\circ$ or when x is of type $\circ$ and y of type $\bullet$) are done in a similar way. $\square$

Proposition 4.3.5. *The map $d : (T_+ \times T_-) \cup (T_- \times T_+) \to \mathbb{N}$ as defined above is a codistance and makes (T_+, T_-, d) into a twin tree.*

Proof. Let $x \in T_+, y \in T_-$ and let $d(x, y) = m$ for some $m \in \mathbb{N}$. It is straightforward (from the way d is defined) that $d(x, y') = m \pm 1$ for all y' adjacent to y in T_-. Now let $m > 0$. We need to check that there exists a unique such y' with $d(x, y') = m+1$. We begin with the existence of y'. Let x' be an adjacent vertex to x, and call its associated chamber $a \in \mathcal{B}_+$. Take any vertex y_1 adjacent to y and call its associated chamber $b_1 \in \mathcal{B}_-$ and the type of y be i and let the type of y_1 be j. If $d(x, y_1) = m+1$ we are done, so suppose it is not. Thus $d(x, y_1)$ must be equal to $m-1$ and then we either have $d(x', y_1) = m$ and $d(x', y) = m+1$ or $d(x', y_1) = m-2$ and $d(x', y) = m-1$.

In the first case (i.e. $d(x', y_1) = m$ and $d(x', y) = m+1$) this means that $\delta(a, b_1)$ has length m and finishes with j, implying that $\delta(b_1, a)$ has length m and begins with i. Now from the definition of δ, we know that there exists a chamber b_2 that is j adjacent to b_1 such that $\delta(b_2, a) = j\delta(b_1, a)$ which means that $\delta(b_2, a)$ has length $m+1$ and starts with j. Say b_2 is associated to the two adjacent vertices y_2—y in T_-. This means that $d(y_2, x') = m+2$ and so $d(x, y_2)$ must be $m+1$.

In the second case (i.e. $d(x', y_1) = m-2$ and $d(x', y) = m-1$) this means that $\delta(a, b_1)$ has length $m-1$ and finishes with i, implying that $\delta(b_1, a)$ has length $m-1$ and begins with i. Again from the definition of δ, we know that there exists a chamber b_2 that is j adjacent to b_1 such that $\delta(b_2, a) = j\delta(b_1, a)$ which means that $\delta(b_2, a)$ has length m and starts with j. Say b_2 is associated to the two adjacent vertices y_2—y in T_-. This means that $d(x, y_2) = m+1$.

We now show uniqueness. Let $x \in T_+, y \in T_-$ and let $d(x, y) = m$ for some $m > 0$. Let y_1, y_2 be adjacent vertices to y in T_- and suppose that $d(x, y_1) = d(x, y_2) = m+1$. Let b_1 be the chamber in $\mathcal{B}_-$ associated to y_1—y, and b_2 the chamber in $\mathcal{B}_-$ associated to y_2—y. Also let x' be any adjacent vertex to x in T_+, and call a the chamber in $\mathcal{B}_+$ that is associated to x—x'.

There are two cases, either $d(x', y_1) = m$ or $d(x', y_1) = m+2$. If it is equal to m then $\delta(a, b_1)$ is of length m, starts with $\circ$ and ends with $\circ$. This implies that $\delta(b_1, a)$ also has length m and starts with $\circ$, but now from the definition of δ and the fact that b_2 is $\circ$-adjacent to b_1, we know that $\delta(b_2, a) = \circ\delta(b_1, a)$ and so

$\delta(b_2, a)$ is of length $m-1$. On the other hand, we also have that $d(x', y) = m-1$ which implies that $\delta(b_2, a)$ is of length m, arriving to a contradiction. The second case is when $d(x', y_1) = m+2$, then $\delta(a, b_1)$ is of length $m+1$ and finishes with $\circ$ implying that $\delta(b_1, a)$ also has length $m+1$ but starts with $\circ$. As before, this implies that $\delta(b_2, a) = \circ\delta(b_1, a)$ which means that $\delta(b_2, a)$ has length m. On the other hand, $d(x', y) = m+1$ and so $\delta(b_2, a)$ has length $m+1$ which is a contradiction. $\square$

CHAPTER 5

PARAHORIC PROJECTION AND THE LIFTING PROBLEM

5.1 Definitions and concepts

We begin with the definition of weakly opposite apartments in the context of twin trees.

Definition 5.1.1 (Weakly opposite apartments)**.** Let (T_+, T_-, d) be a twin tree and fix a vertex y in T_-. We say that the apartment A in T_+ is *weakly opposite* to y if for all vertices $x \in A$, we have that $d(x, y)$ is either 0 or 1.

In this chapter we use the same notations as the ones we had in Section 4.2.3. Thus we have the twin tree $(\mathcal{T}_0, \mathcal{T}_\infty, d)$ (constructed in Section 4.2.3) and let $(\mathcal{B}_0, \mathcal{B}_\infty, \delta)$ be its associated twin tree in the building sense as explained in Section 4.3.

Given a tree, a *half-apartment* in it is a path without repeated edges and with exactly one end point (i.e. infinite in one direction).

Lemma 5.1.2. *[35, page 472] Half-apartments in T_- come in two types:*

(1) Those where the codistance to any vertex in T_+ is 0 *finitely many times after which point it will increase monotonically.*

(2) Those where the codistance to any vertex in T_+ remains bounded and hits 0 *infinitely many times.*

5.2 Parahoric projection, the lifting problem and existence of weakly opposite apartments

Fix a vertex $y \in \mathcal{T}_\infty$ and let $\mathcal{T}_{0,y}$ be the subgraph of $\mathcal{T}_0$ with vertices at codistance 1 from y.

Let $\mathscr{A}_y : \mathbb{Z} \to \mathcal{T}_{0,y}$ be a map such that $\mathscr{A}_y(m) \neq \mathscr{A}_y(m+1)$ for all $m \in \mathbb{Z}$ and such that there exist distinct x_m and x_{m+1} in $\mathcal{T}_0$ that are at codistance equal to 0 from y with the property that x_m is adjacent to both $\mathscr{A}_y(m)$ and $\mathscr{A}_y(m+1)$ and x_{m+1} is adjacent to $\mathscr{A}_y(m+1)$ and $\mathscr{A}_y(m+2)$. We call $\mathscr{A}_y$ a *labelled apartment*. Clearly, we can complete the image of $\mathbb{Z}$ under $\mathscr{A}_y$ with all of the vertices $x_m (m \in \mathbb{Z})$ that are at codistance equal to 0 from y to form a weakly opposite apartment A_y to y. This apartment A_y is what we will refer to as *the associated apartment* to the labelled apartment $\mathscr{A}_y$.

Similarly we say that $\mathscr{A}_y^+ : \mathbb{N} \to \mathcal{T}_{0,y}$ (respectively $\mathscr{A}_y^- : \mathbb{Z} \setminus \mathbb{N}^* \to \mathcal{T}_{0,y}$) is a *labelled half-apartment* if $\mathscr{A}_y(m) \neq \mathscr{A}_y(m+1)$ for all $m \in \mathbb{N}$ (respectively, for all $m \in \mathbb{Z} \setminus \mathbb{N}$) and such that there exist distinct x_m and x_{m+1} in $\mathcal{T}_0$ that are at codistance equal to 0 from y with the property that x_m is adjacent to both $\mathscr{A}_y(m)$ and $\mathscr{A}_y(m+1)$ and x_{m+1} is adjacent to $\mathscr{A}_y(m+1)$ and $\mathscr{A}_y(m+2)$.

Now let l_y be the set of all adjacent vertices to y in $\mathcal{T}_\infty$.

Definition 5.2.1 (Parahoric projection)**.** We say that the map $\Pi_y : \mathcal{T}_{0,y} \to l_y$ is the *parahoric projection* from y if for all $\hat{x} \in \mathcal{T}_{0,y}$, $\Pi_y(\hat{x})$ is the unique element in l_y such that $d(\hat{x}, \Pi_y(\hat{x})) = 2$. We say that $\Pi_y(\hat{x})$ is the *parahoric projection* of $\hat{x}$ onto l_y from y.

Given a map $g_y : \mathbb{Z} \to l_y$, a *lift* is a labelled apartment $\mathscr{A}_y : \mathbb{Z} \to \mathcal{T}_{0,y}$ such that $\Pi_y \circ \mathscr{A}_y = g_y$. Now let A_y be the associated apartment to $\mathscr{A}_y$. In this case, we say that the apartment A_y *parahorically projects* onto g_y from y.

Before we proceed with the following lemma and theorem, let us say that the maps $g_y : \mathbb{Z} \to \mathcal{T}_{0,y}$ and $g_y^+ : \mathbb{N} \to \mathcal{T}_{0,y}$ are *without repetition* if $g_y(m) \neq g_y(m+1)$ for all $m \in \mathbb{Z}$, and respectively $g_y^+(m) \neq g_y^+(m+1)$ for all $m \in \mathbb{N}$. We also say that $g_y^- : \mathbb{Z} \setminus \mathbb{N}^* \to \mathcal{T}_{0,y}$ is *without repetition* if $g_y^-(m) \neq g_y^-(m-1)$ for all $m \in \mathbb{Z} \setminus \mathbb{N}^*$.

Lemma 5.2.2. *Let $\widehat{x}_0, \widehat{x}_1 \in \mathcal{T}_{0,y}$ such that there exists a vertex $x \in \mathcal{T}_0$ that is adjacent to both $\widehat{x}_0$ and $\widehat{x}_1$ and such that $d(x, y) = 0$. Given a map $g_y^+ : \mathbb{N} \to l_y$ without repetition and with the conditions that $g_y^+(0) = \Pi_y(\widehat{x}_0)$ and $g_y^+(1) =$*

$\Pi_y(\widehat{x}_1)$, *then we can find a labelled half-apartment* $\mathscr{A}_y^+ : \mathbb{N} \to \mathcal{T}_{0,y}$ *such that* $\Pi_y(\mathscr{A}_y^+) = g_y^+$. *Similarly, given a map* $g_y^- : \mathbb{Z} \setminus \mathbb{N}^* \to l_y$ *without repetition, with the conditions that* $g_y^-(0) = \Pi_y(\widehat{x}_0)$ *and* $g_y^-(-1) = \Pi_y(\widehat{x}_1)$, *then we can find a labelled half-apartment* $\mathscr{A}_y^- : \mathbb{Z} \setminus \mathbb{N}^* \to \mathcal{T}_{0,y}$ *such that* $\Pi_y(\mathscr{A}_y^-) = g_y^-$.

Proof. By symmetry we only need to define $\mathscr{A}_{y,n}^+$, the map $\mathscr{A}_{y,n}^-$ is constructed in a similar way. We begin by defining a map $\mathscr{A}_{y,n}^+ : [\![0, n]\!] \to \mathcal{T}_{0,y}$ for all natural numbers $n \geqslant 2$ by induction, that satisfies the following condition. Firstly, $\mathscr{A}_{y,n}^+(m) \neq \mathscr{A}_{y,n}^+(m+1)$ for all $m \in [\![0, n-1]\!]$. Then for all $m \in [\![0, n-2]\!]$ there exist distinct x_m and x_{m+1} in $\mathcal{T}_0$ that are at codistance equal to 0 from y with the property that x_m is adjacent to both $\mathscr{A}_{y,n}^+(m)$ and $\mathscr{A}_{y,n}^+(m+1)$ and x_{m+1} is adjacent to $\mathscr{A}_{y,n}^+(m+1)$ and $\mathscr{A}_{y,n}^+(m+2)$. Finally, we want $\Pi_y(\mathscr{A}_{y,n}^+) = g_{y|[\![0,n]\!]}^+$.

Let $\mathscr{A}_{y,1}^+ : [\![0, 2]\!] \to \mathcal{T}_{0,y}$ be such that $\mathscr{A}_{y,1}^+(0) := \Pi_y(\widehat{x}_0)$ and $\mathscr{A}_{y,1}^+(1) := \Pi_y(\widehat{x}_1)$. Now there exists a vertex $x' \in \mathcal{T}_0$ that is adjacent to $\widehat{x}_1$ such that $d(x', y) = 0$ and $x' \neq x$. Let $\mathscr{A}_{y,2}^+(2)$ be the unique adjacent vertex to x' that is at codistance from $g_y^+(2)$ equal to 2. Note that $\mathscr{A}_{y,2}^+(2) \neq \widehat{x}_1$ since $d(\widehat{x}_1, g_y^+(2)) = 0$.

Now suppose we have such a map $\mathscr{A}_{y,n-1}^+ : [\![0, n-1]\!] \to \mathcal{T}_{0,y}$ and we construct the map $\mathscr{A}_{y,n}^+ : [\![0, n]\!] \to \mathcal{T}_{0,y}$. Let $\mathscr{A}_{y,n|[\![0,n-1]\!]}^+ = \mathscr{A}_{y,n-1}^+$, and so we only need to define $\mathscr{A}_{y,n}^+(n)$. Since $d(\mathscr{A}_{y,n}^+(n-1), y) = 1$, pick any adjacent x_{n-1} to $\mathscr{A}_{y,n}^+(n-1)$ that is at codistance equal to 0 from y and such that $x_{n-1} \neq x_{n-2}$. Now let $\mathscr{A}_{y,n}^+(n)$ be the unique vertex that is adjacent to x_{n-1} and is at codistance equal to 2 from $g_y^+(n)$. Note that $\mathscr{A}_{y,n}^+(n) \neq \mathscr{A}_{y,n}^+(n-1)$ since $d(g_y^+(n), \mathscr{A}_{y,n}^+(n-1)) = 0$.

Finally, define $\mathscr{A}_y^+ : \mathbb{N} \to \mathcal{T}_{0,y}$ as follows: $\mathscr{A}_y^+ := \cup_{n \geqslant 2} \mathscr{A}_{y,n}^+$. This is a labelled half-apartment that satisfies $\Pi_y(\mathscr{A}_y^+) = g_y^+$. □

Theorem 5.2.3. *Let* $g_y : \mathbb{Z} \to l_y$ *be without repetition. Then* g_y *has a lift.*

Proof. Let $x_0 \in \mathcal{T}_0$ such that $d(x_0, y) = 0$. This implies that $d(x_0, g_y(0)) = d(x_0, g_y(1)) = 1$, and so there exist a unique adjacent vertex $\widehat{x}_0 \in \mathcal{T}_{0,y}$ to x_0 such that $d(\widehat{x}_0, g_y(0)) = 2$ and a unique adjacent vertex $\widehat{x}_1 \in \mathcal{T}_{0,y}$ to x_0 such that $d(\widehat{x}_1, g_y(1)) = 2$. Note that $\widehat{x}_0 \neq \widehat{x}_1$ since $d(g_y(1), \widehat{x}_0) = 0$ ($g_y(0) \neq g_y(1)$ because g_y is without repetition). Now from Lemma 5.2.2 we have a labelled half-apartment $\mathscr{A}_y^+ : \mathbb{N} \to \mathcal{T}_{0,y}$ such that $\Pi_y(\mathscr{A}_y^+) = g_{y|\mathbb{N}}$.

Similarly, pick an adjacent vertex x_{-1} to $\widehat{x}_0$, with $x_{-1} \neq x_0$, such that $d(x_{-1}, y) = 0$. This is possible since there are at least two distinct adjacent vertices to $\widehat{x}_0$ that are at codistance equal to 0 from y. Now we have $d(g_y(-1), x_{-1}) = 1$ and so there exists a unique $\widehat{x}_{-1}$ adjacent to x_{-1} that is at codistance equal to 2 from $g_y(-1)$, and $\widehat{x}_{-1} \neq \widehat{x}_0$ since $d(g_y(-1), \widehat{x}_0) = 0$ ($g_y(-1) \neq g_y(0)$ because g_y is without repetition). Again from Lemma 5.2.2 we have a labelled half-apartment $\mathscr{A}_y^- : \mathbb{Z} \setminus \mathbb{N}^* \to \mathcal{T}_{0,y}$ such that $\Pi_y(\mathscr{A}_y^-) = g_{y|\mathbb{Z}\setminus\mathbb{N}^*}$.

Now define the map $\mathscr{A}_y : \mathbb{Z} \to \mathcal{T}_{0,y}$ as follows: $\mathscr{A}_{y|\mathbb{N}} := \mathscr{A}_y^+$ and $\mathscr{A}_{y|\mathbb{Z}\setminus\mathbb{N}^*} := \mathscr{A}_y^-$. Now since $x_0 \neq x_{-1}$ and since $\mathscr{A}_y^+$ and $\mathscr{A}_y^-$ are labelled half-apartments we have that $\mathscr{A}_y$ is a labelled apartment. Finally $\Pi_y(\mathscr{A}_y) = g_y$ and so $\mathscr{A}_y$ is a lift for g_y. ☐

Corollary 5.2.4. *There is a weakly opposite apartment to y in $\mathcal{T}_0$.*

Proof. Pick y_1 and y_2 two distinct adjacent vertices to y, and define the map without repetition $g_y : \mathbb{Z} \to l_y$ such that $g_y(\mathbb{Z}) = \{y_1, y_2\}$ and $g_y(0) = y_1$. From Theorem 5.2.3, we have a lift $\mathscr{A}_y : \mathbb{Z} \to \mathcal{T}_{0,y}$. Therefore taking the image of $\mathscr{A}_y$ and adding the adjacent vertices that are at codistance 0 from y to make it into an apartment in $\mathcal{T}_0$, we obtain a weakly opposite apartment to y. ☐

5.3 An example of a lifting problem solution

As an illustration of the previous section we give an example of a map $g_y : \mathbb{Z} \mapsto l_y$ for a fixed $y \in \mathcal{T}_\infty$, and find a specific apartment A_x in $\mathcal{T}_0$ that is weakly opposite to y and that also parahorically projects onto g_y from y.

Fix the lattice $Y = \mathbb{k}[t^{-1}]e_1 \oplus \mathbb{k}[t^{-1}]e_2$ and the vertex $y = [Y] \in \mathcal{T}_\infty$ and let l_y be the set of all adjacent vertices to y in $\mathcal{T}_\infty$. Let $Y_1 := [te_1, e_2]$ and $Y_2 := [e_1, te_2]$ and note that $y_1 := [Y_1]$ and $y_2 := [Y_2]$ in $\mathcal{T}_\infty$ are two vertices that are adjacent to y, and so they are in l_y. Let $g_y : \mathbb{Z} \mapsto l_y$ be a map without repetition with its image being the set $\{y_1, y_2\}$ (i.e. either g_y is the map that sends 1 and 2 to y_1 and y_2 or to y_2 and y_1 respectively).

In this section, we construct an apartment A_x in $\mathcal{T}_0$ and show that it is weakly opposite to y, in Theorem 5.3.1. We also prove that it parahorically projects onto g_y from y, in Theorem 5.3.2.

Let $x := \left[\mathbb{k}[t]e_1 \oplus \mathbb{k}[t]e_2\right] \in \mathcal{T}_0$. Let us begin by showing that $d(x, y) = 0$. We have $X \cap Y = \mathbb{k}e_1 \oplus \mathbb{k}e_2$ and so $e_1, e_2 \in X \cap Y$, thus $X \cap Y$ contains a basis of $\mathbb{K}^2$, while $tX \cap Y = \{0\}$, and so $tX \cap Y$ does not contain a basis of $\mathbb{K}^2$. Therefore, by definition, $d(x, y) := \dim_{\mathbb{k}}(tX \cap Y) = 0$.

We would like now to construct an apartment A_x in $\mathcal{T}_0$ around x such that it is weakly opposite to y. Using the symmetry in e_1 and e_2, it will suffice to construct a half-apartment: the other half (and other end) is obtained by swapping the roles of e_1 and e_2.

Let the vertices of A_x be of the form

$$x_{m,n} := \left[\left[e_1 - t^{-1}\left(\sum_{i=1}^{m} t^{2^i}\right) e_2, t^{2^m+n} e_2\right]\right],$$

with integers $m \geqslant 0$ and $0 \leqslant n \leqslant 2^m - 1$. In order to see that this indeed forms a half-apartment, we show that for any of these vertices, the next one is indeed incident to it. Let $x_{m,n}$ be any vertex, then the next vertex is $x_{m+1,0}$ if $n = 2^m - 1$, and $x_{m,n+1}$ otherwise. We want to show that this next vertex is incident with $x_{m,n}$.

Let us begin with the case $n = 2^m - 1$. We have

$$\begin{aligned} x_{m,2^m-1} &= \left[\left[e_1 - t^{-1}\left(\sum_{i=1}^{m} t^{2^i}\right) e_2, t^{2^{m+1}-1} e_2\right]\right] \\ &= \left[\left[e_1 - t^{-1}\left(\sum_{i=1}^{m} t^{2^i}\right) e_2 - t^{2^{m+1}-1} e_2, t^{2^{m+1}-1} e_2\right]\right] \\ &= \left[\left[e_1 - t^{-1}\left(\sum_{i=1}^{m+1} t^{2^i}\right) e_2, t^{2^{m+1}-1} e_2\right]\right]. \end{aligned}$$

So let us pick $\lambda_{m,2^m-1} := \left[e_1 - t^{-1}\left(\sum_{i=1}^{m+1} t^{2^i}\right) e_2, t^{2^{m+1}-1} e_2\right]$ $\left(\text{so that } x_{m,2^m-1} = [\lambda_{m,2^m-1}]\right)$. We also have $x_{m+1,0} = \left[\left[e_1 - t^{-1}\left(\sum_{i=1}^{m+1} t^{2^i}\right) e_2, t^{2^{m+1}} e_2\right]\right]$ so let us pick $\lambda_{m+1,0} = \left[e_1 - t^{-1}\left(\sum_{i=1}^{m+1} t^{2^i}\right) e_2, t^{2^{m+1}} e_2\right]$ $\left(\text{so that } x_{m+1,0} = \lambda_{m+1,0}\right)$.

Now notice that $\lambda_{m+1,0} \lneqq \lambda_{m,2^m-1}$, and that

$$t\lambda_{m,2^m-1} = \left[t\left(e_1 - t^{-1}\left(\sum_{i=1}^{m+1} t^{2^i}\right) e_2\right), t^{2^{m+1}} e_2\right] \lneqq \lambda_{m+1,0},$$

therefore $x_{m,2^m-1}$ and $x_{m+1,0}$ are indeed incident.

Now in the case of $n \neq 2^m - 1$. We have $x_{m,n} := \left[\left[e_1 - t^{-1}\left(\sum_{i=1}^{m} t^{2^i}\right) e_2, t^{2^m+n} e_2\right]\right]$ and we pick $\lambda_{m,n} = \left[e_1 - t^{-1}\left(\sum_{i=1}^{m} t^{2^i}\right) e_2, t^{2^m+n} e_2\right]$ $\left(\text{so that } x_{m,n} = [\lambda_{m,n}]\right)$. We also have $x_{m,n+1} := \left[\left[e_1 - t^{-1}\left(\sum_{i=1}^{m} t^{2^i}\right) e_2, t^{2^m+n+1} e_2\right]\right]$ and so we pick $\lambda_{m,n+1} = \left[e_1 - t^{-1}\left(\sum_{i=1}^{m} t^{2^i}\right) e_2, t^{2^m+n+1} e_2\right]$ $\left(\text{where } x_{m,n+1} = [\lambda_{m,n+1}]\right)$. Clearly $\lambda_{m,n+1} \lneqq$

$\lambda_{m,n}$. We also have that

$$t\lambda_{m,n} = \left[t\left(e_1 - t^{-1}\left(\sum_{i=1}^{m} t^{2^i} \right) e_2 \right), t^{2^m+n+1} e_2 \right] \lneqq \lambda_{m,n+1},$$

therefore $x_{m,n}$ and $x_{m,n+1}$ are incident. This half-apartment is illustrated in Figure 5-1 below.

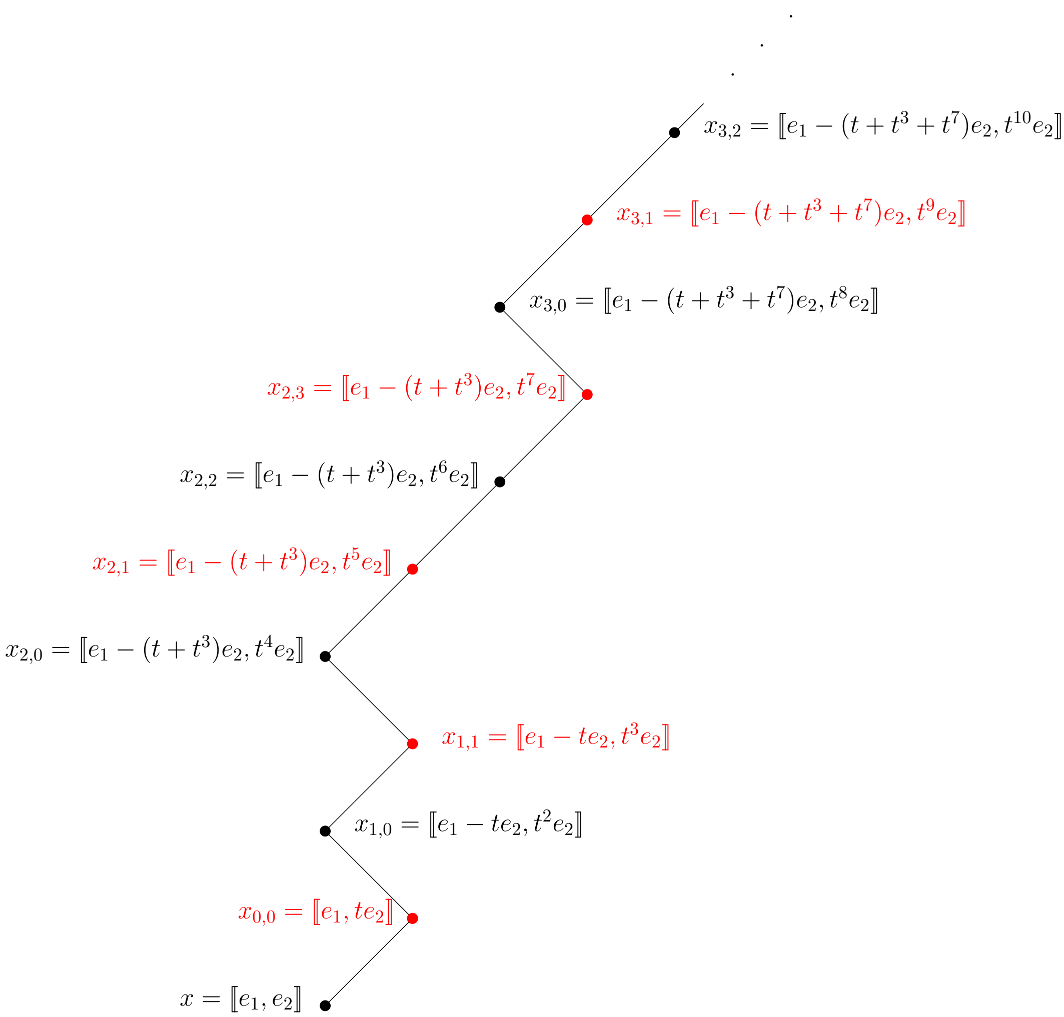

Figure 5-1

As previously stated, in this section we prove the following two theorems.

Theorem 5.3.1. *The apartment A_x is weakly opposite to y.*

Theorem 5.3.2. *The apartment A_x parahorically projects onto g_y.*

5.3.1 Proof of Theorem 5.3.1

In the following subsection we prove that for all $m \geqslant 2$ and $0 \leqslant k \leqslant 2^{m-1} - 1$ the codistance $d(x_{m,2k}, y) = 0$. Thus by the definition of the codistance $d(x_{m,2k+1}, y) = 1$ and so we can conclude that the apartment A_x is weakly opposite to y, proving Theorem 5.3.1.

We start by introducing some definitions that we will use in the proof of this theorem. Let $X_{m,2k} = [X_{m,2k}(1), X_{m,2k}(2)]$ be a $\Bbbk[t]$-lattice in $\mathbb{K}^2$ where $X_{m,2k}(1) := t^{-(2^{m-1}+k)}e_1 - t^{-(2^{m-1}+k+1)}\left(\sum_{i=1}^{m} t^{2^i}\right)e_2$ and $X_{m,2k}(2) := t^{2^{m-1}+k}e_2$. Thus $x_{m,2k} = [X_{m,2k}] \in \mathcal{T}_0$. We also introduce two elements $a_{m,2k}$ and $b_{m,2k} \in \mathbb{K}^2$, that will depend on some signs $\delta_{m,2(k-1)} \in \{-1, +1\}$ that we will specify later. We write

$$a_{m,2k} = a_{m,2k}(1)X_{m,2k}(1) + a_{m,2k}(2)X_{m,2k}(2), \text{ and}$$

$$b_{m,2k} = b_{m,2k}(1)X_{m,2k}(1) + b_{m,2k}(2)X_{m,2k}(2)$$

with respect to the basis $X_{m,2k}(1), X_{m,2k}(2)$ of $\mathbb{K}^2$ and define them, for any $m \geqslant 2$ and $0 \leqslant k \leqslant 2^{m-1} - 1$, recursively as follows:

$$a_{m,0}(1) := 1 + \left(\sum_{i=1}^{m-2} t^{2^i}\right) - t^{2^{m-1}}, a_{m,0}(2) := t^{-1}\left(\sum_{i=1}^{m-2} t^{2^i}\right) - t^{2^{m-1}-1},$$

$$b_{m,0}(1) := -t, b_{m,0}(2) := -1,$$

$$a_{m,2k} := t^{-1}a_{m,2(k-1)} + b_{m,2(k-1)}, \text{ and}$$

$$b_{m,2k} := \delta_{m,2(k-1)}a_{m,2(k-1)}.$$

Lemma 5.3.3. *Let $m \geqslant 2$ and $1 \leqslant k \leqslant 2^{m-1} - 1$. The following recursive formulas hold: $X_{m,2k}(1) = t^{-1}X_{m,2(k-1)}(1)$ and $X_{m,2k}(2) = tX_{m,2(k-1)}(2)$, as well as*

$$a_{m,2k}(1) = a_{m,2(k-1)}(1) + tb_{m,2(k-1)}(1), \quad a_{m,2k}(2) = t^{-2}a_{m,2(k-1)}(2) + t^{-1}b_{m,2(k-1)}(2),$$
$$b_{m,2k}(1) = \delta_{m,2(k-1)}ta_{m,2(k-1)}(1), \qquad b_{m,2k}(2) = \delta_{m,2(k-1)}t^{-1}a_{m,2(k-1)}(2).$$

In particular, all the powers of t in the Laurent polynomials $a_{m,2k}(2)$ and $b_{m,2k}(1)$ are odd and all the powers of t in the Laurent polynomials $a_{m,2k}(1)$ and $b_{m,2k}(2)$ are even.

Proof. Firstly we have,

$$\begin{aligned}
X_{m,2k}(1) &= t^{-(2^{m-1}+k)}e_1 - t^{-(2^{m-1}+k+1)}\left(\sum_{i=1}^{m} t^{2^i}\right)e_2 \\
&= t^{-1}.t^{-(2^{m-1}+k-1)}e_1 - t^{-1}.t^{-(2^{m-1}+(k-1)+1)}\left(\sum_{i=1}^{m} t^{2^i}\right)e_2 \\
&= t^{-1}X_{m,2(k-1)}(1), \text{ as well as}
\end{aligned}$$

$$\begin{aligned}
X_{m,2k}(2) &= t^{2^{m-1}+k}e_2 \\
&= t.t^{2^{m-1}+(k-1)}e_2 \\
&= tX_{m,2(k-1)}(2).
\end{aligned}$$

Now, from the definition of $a_{m,2k}$, we have

$$\begin{aligned}
a_{m,2k} &= t^{-1}a_{m,2(k-1)} + b_{m,2(k-1)} \\
&= t^{-1}\left(a_{m,2(k-1)}(1)X_{m,2(k-1)}(1) + a_{m,2(k-1)}(2)X_{m,2(k-1)}(2)\right) \\
&+ \left(b_{m,2(k-1)}(1)X_{m,2(k-1)}(1) + b_{m,2(k-1)}(2)X_{m,2(k-1)}(2)\right) \\
&= \left(t^{-1}a_{m,2(k-1)}(1) + b_{m,2(k-1)}(1)\right)X_{m,2(k-1)}(1) \\
&+ \left(t^{-1}a_{m,2(k-1)}(2) + b_{m,2(k-1)}(2)\right)X_{m,2(k-1)}(2).
\end{aligned}$$

And so replacing with the previous two equations we get

$$\begin{aligned}
a_{m,2k} &= \left(t^{-1}a_{m,2(k-1)}(1) + b_{m,2(k-1)}(1)\right)(tX_{m,2k}(1)) \\
&+ \left(t^{-1}a_{m,2(k-1)}(2) + b_{m,2(k-1)}(2)\right)\left(t^{-1}X_{m,2k}(2)\right) \\
&= \left(a_{m,2(k-1)}(1) + tb_{m,2(k-1)}(1)\right)X_{m,2k}(1) \\
&+ \left(t^{-2}a_{m,2(k-1)}(2) + t^{-1}b_{m,2(k-1)}(2)\right)X_{m,2k}(2).
\end{aligned}$$

Therefore, we conclude that $a_{m,2k}(1) = a_{m,2(k-1)}(1) + tb_{m,2(k-1)}(1)$ and $a_{m,2k}(2) = t^{-2}a_{m,2(k-1)}(2) + t^{-1}b_{m,2(k-1)}(2)$. Finally, we obtain the expressions for $b_{m,2k}(1)$ and $b_{m,2k}(2)$ in a similar way. As for the parity claims, they hold for $k = 1$ and the general case is an easy induction on k using the equations above. □

We now associate to every m a diagram where the vertical edges are labelled by $\delta_{m,2k}$ for $0 \leqslant k \leqslant 2^{m-1}-3$. Once we have constructed the diagram associated to m, we can easily read the value of $\delta_{m,2k}$ from it.

First note that for all $2 \leqslant k \leqslant 2^{m-1}-1$, by Lemma 5.3.3, $a_{m,2k}(2) = t^{-2}a_{m,2(k-1)}(2) + t^{-1}b_{m,2(k-1)}(2)$ and $b_{m,2(k-1)}(2) = \delta_{m,2(k-2)}t^{-1}a_{m,2(k-2)}(2)$.

Thus,

$$a_{m,2k}(2) = t^{-2}a_{m,2(k-1)}(2) + \delta_{m,2(k-2)}t^{-2}a_{m,2(k-2)}(2). \tag{5.3.1}$$

This means that for $2 \leqslant k \leqslant 2^{m-1}-1$ we obtain $a_{m,2k}(2)$ from $a_{m,2(k-1)}(2)$ and $a_{m,2(k-2)}(2)$ and from knowing the value of $\delta_{m,2(k-2)}$.

The recursive formulae 5.3.1 at level m can be illustrated by the diagram in Figure 5-2, which we read from top to bottom as follows. The value at each vertex is the sum of the value at the vertex directly above multiplied by t^{-2} and the value at the vertex diagonally above it multiplied by $\delta_{m,2k}t^{-2}$. So as an illustration, $a_{m,2(k+2)}(2) = t^{-2}a_{m,2(k+1)}(2) + \delta_{m,2k}t^{-2}a_{m,2k}(2)$. An exception is for $a_{m,2}(2)$ (i.e. for $k=1$) where $a_{m,2}(2) = t^{-2}a_{m,0}(2) + t^{-1}b_{m,0}(2)$ and so in the diagram, $b_{m,0}(2)$ is multiplied by t^{-1} and not by t^{-2}.

It is now possible to define these diagrams independently in order to use them for our proof. We recursively define the diagram associated to a fixed m as follows. For $m=3$, the diagram is as shown in Figure 5-3 below.

Then the diagram associated to m is obtained by first taking the diagram of $m-1$ (without the part with $b_{m-1,0}(2)$), rotating it and adding the part with $b_{m,0}(2)$ at the top left, then adding at the bottom two edges labelled with $\delta_{m,2(2^{m-2}-2)} := -1$ on the right and $\delta_{m,2(2^{m-2}-1)} := +1$ on the left, and finally adding at the bottom a copy of the diagram of $m-1$ (again without the part with $b_{m-1,0}(2)$). See Figures 5-4 and 5-5.

Let us call L_m this diagram associated to m and let U_m be the diagram obtained from L_m by applying a π rotation (about the central zag). See Figure 5-6 below for an illustration of this.

Proposition 5.3.4. *Let us define* $S_m := \sum_{i=1}^{m-2} t^{2^i}$ *and* $\widetilde{S}_m := \left(\sum_{i=1}^{m-3} t^{2^i}\right) - t^{2^{m-2}}$. *With notation from Figure 5-6, we have the following formulas for* L_m*:*

$$x'_{L,m} = t^{-2^m}\left(t^2 S_{m+1} - S_m^2\right)x_{L,m} + t^{-2^m+2}S_m^2 y_{L,m}, \tag{5.3.2}$$

$$y'_{L,m} = t^{-2^m}\left(t^2 - \widetilde{S}_{m+1}\right)x_{L,m} + t^{-2^m+2}\widetilde{S}_{m+1}y_{L,m}. \tag{5.3.3}$$

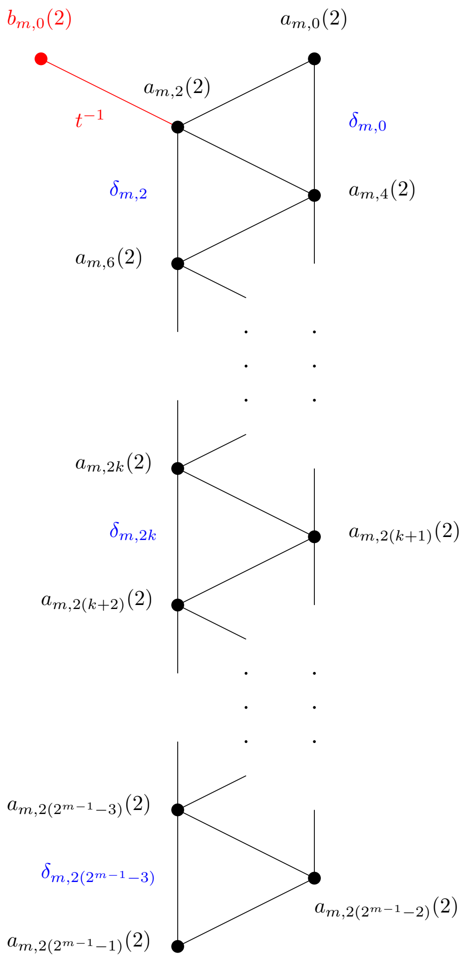

Figure 5-2

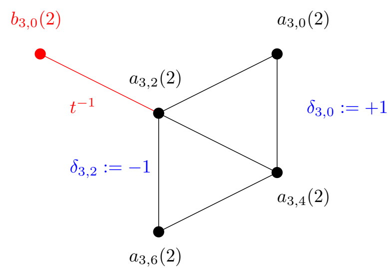

Figure 5-3: *Diagram for $m = 3$*

We also have the following for U_m:

$$x'_{U,m} = t^{-2^m} \left(t^2 \widetilde{S}_{m+1} - S_m^2 \right) x_{U,m} + t^{-2^m+2} S_m^2 y_{U,m}, \tag{5.3.4}$$

$$y'_{U,m} = t^{-2^m} \left(t^2 - S_{m+1} \right) x_{U,m} + t^{-2^m+2} S_{m+1} y_{U,m}. \tag{5.3.5}$$

Proof. We prove this proposition by induction. It is straightforward to check that

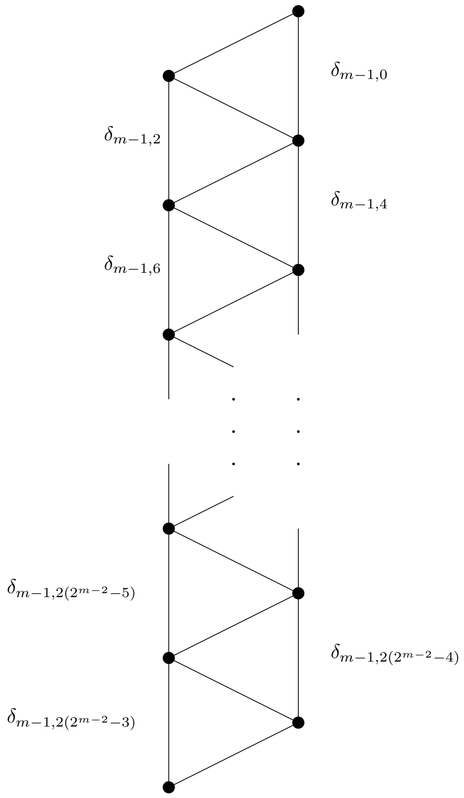

Figure 5-4: *Diagram for $m-1$ (without the top left edge)*

the equations are correct for $m = 4$. Now suppose they are true for m, we want to show that they also hold for $m+1$.

We know that the bottom halves of L_{m+1} and U_{m+1} are copies of L_m (see Figure 5-7), and so this means (using the induction hypothesis) that:

$$x'_{L,m+1} = t^{-2^m}\left(t^2 S_{m+1} - S_m^2\right)\hat{x}_{L,m+1} + t^{-2^m+2}S_m^2\hat{y}_{L,m+1}, \tag{5.3.6}$$

$$y'_{L,m+1} = t^{-2^m}\left(t^2 - \widetilde{S}_{m+1}\right)\hat{x}_{L,m+1} + t^{-2^m+2}\widetilde{S}_{m+1}\hat{y}_{L,m+1}, \tag{5.3.7}$$

$$x'_{U,m+1} = t^{-2^m}\left(t^2\widetilde{S}_{m+1} - S_m^2\right)\hat{x}_{U,m+1} + t^{-2^m+2}S_m^2\hat{y}_{U,m+1}, \tag{5.3.8}$$

$$y'_{U,m+1} = t^{-2^m}\left(t^2 - S_{m+1}\right)\hat{x}_{U,m+1} + t^{-2^m+2}S_{m+1}\hat{y}_{U,m+1}. \tag{5.3.9}$$

Now note that $\hat{x}_{L,m+1} = -t^{-2}\dot{x}_{L,m+1} + t^{-2}\dot{y}_{L,m+1}$ and $\hat{y}_{L,m+1} = t^{-2}\hat{x}_{L,m+1} + t^{-2}\dot{y}_{L,m+1}$. So $\hat{y}_{L,m+1} = t^{-2}\left(-t^{-2}\dot{x}_{L,m+1} + t^{-2}\dot{y}_{L,m+1}\right) + t^{-2}\dot{y}_{L,m+1} = -t^{-4}\dot{x}_{L,m+1} + \left(t^{-4} + t^{-2}\right)\dot{y}_{L,m+1}$. We also have $\hat{x}_{U,m+1} = t^{-2}\dot{x}_{U,m+1} + t^{-2}\dot{y}_{U,m+1}$ and $\hat{y}_{U,m+1} = t^{-2}\hat{x}_{U,m+1} - t^{-2}\dot{y}_{U,m+1}$, and so $\hat{y}_{U,m+1} = t^{-2}\left(t^{-2}\dot{x}_{U,m+1} + t^{-2}\dot{y}_{U,m+1}\right) - t^{-2}\dot{y}_{U,m+1} = t^{-4}\dot{x}_{U,m+1} + (t^{-4} - t^{-2})\dot{y}_{U,m+1}$.

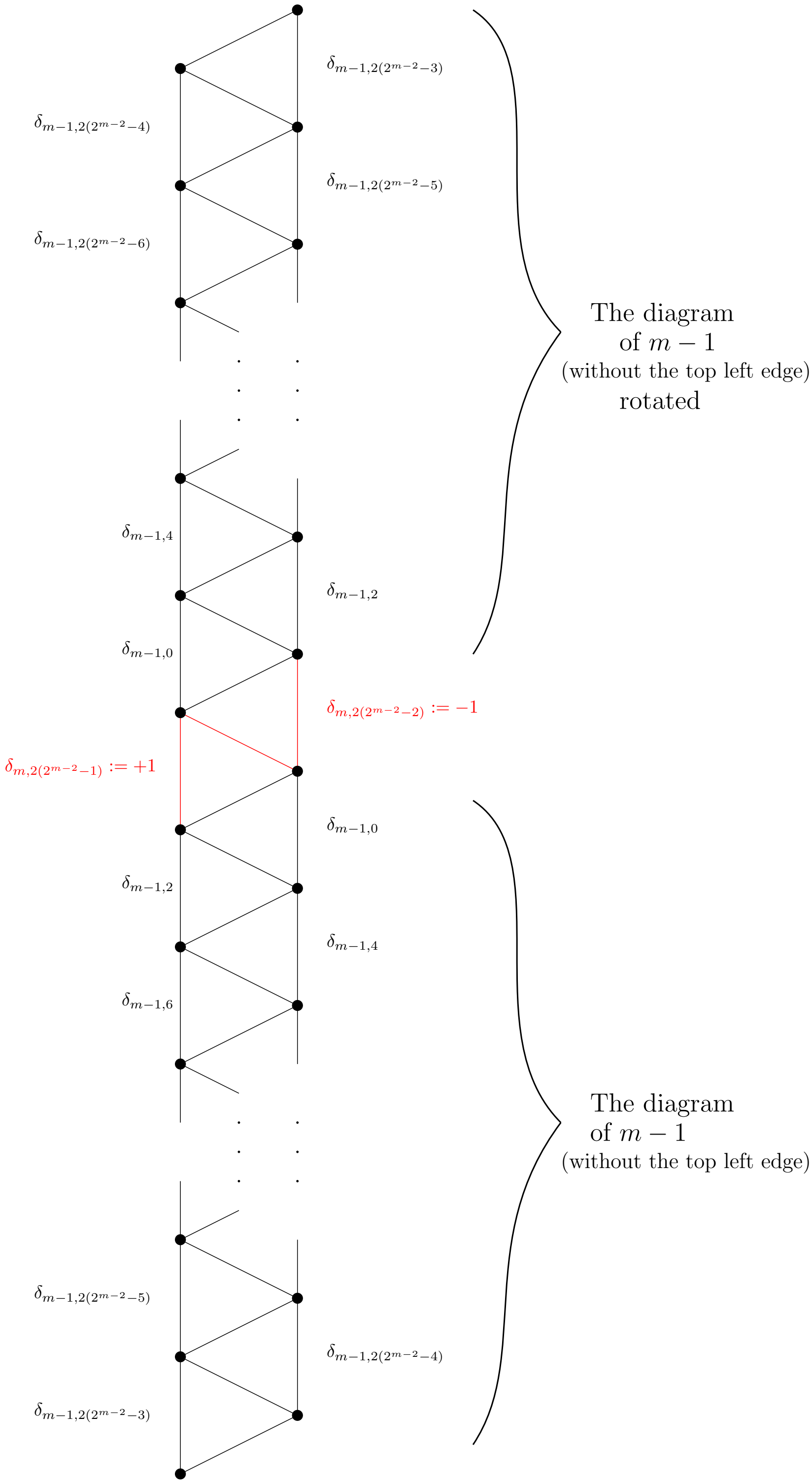

Figure 5-5: *Diagram for m (without the top left edge)*

Therefore after replacing in Equations (5.3.6), (5.3.7), (5.3.8) and (5.3.9), we

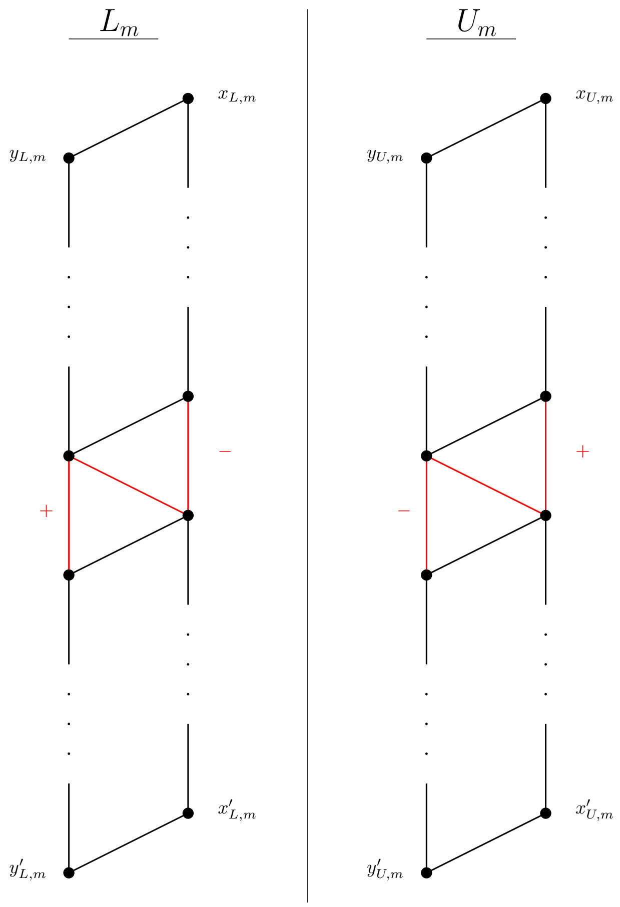

Figure 5-6

obtain the following.

$$x'_{L,m+1} = -t^{-2^m} S_{m+1}\dot{x}_{L,m+1} + t^{-2^m}\left(S_{m+1} + S_m^2\right)\dot{y}_{L,m+1}, \tag{5.3.10}$$

$$y'_{L,m+1} = -t^{-2^m}\dot{x}_{L,m+1} + t^{-2^m}\left(1 + \widetilde{S}_{m+1}\right)\dot{y}_{L,m+1}, \tag{5.3.11}$$

$$x'_{U,m+1} = t^{-2^m} S_{m+1}\dot{x}_{U,m+1} + t^{-2^m}\left(S_{m+1} - S_m^2\right)\dot{y}_{U,m+1}, \tag{5.3.12}$$

$$y'_{U,m+1} = t^{-2^m}\dot{x}_{U,m+1} + t^{-2^m}\left(1 - \widetilde{S}_{m+1}\right)\dot{y}_{U,m+1}. \tag{5.3.13}$$

Finally, since the top halves of L_{m+1} and U_{m+1} are copies of U_m, we also have:

$$\dot{x}_{L,m+1} = t^{-2^m}\left(t^2\widetilde{S}_{m+1} - S_m^2\right)x_{L,m+1} + t^{-2^m+2}S_m^2 y_{L,m+1},$$

$$\dot{y}_{L,m+1} = t^{-2^m}\left(t^2 - S_{m+1}\right)x_{L,m+1} + t^{-2^m+2}S_{m+1}y_{L,m+1},$$

$$\dot{x}_{U,m+1} = t^{-2^m}\left(t^2\widetilde{S}_{m+1} - S_m^2\right)x_{U,m+1} + t^{-2^m+2}S_m^2 y_{U,m+1},$$

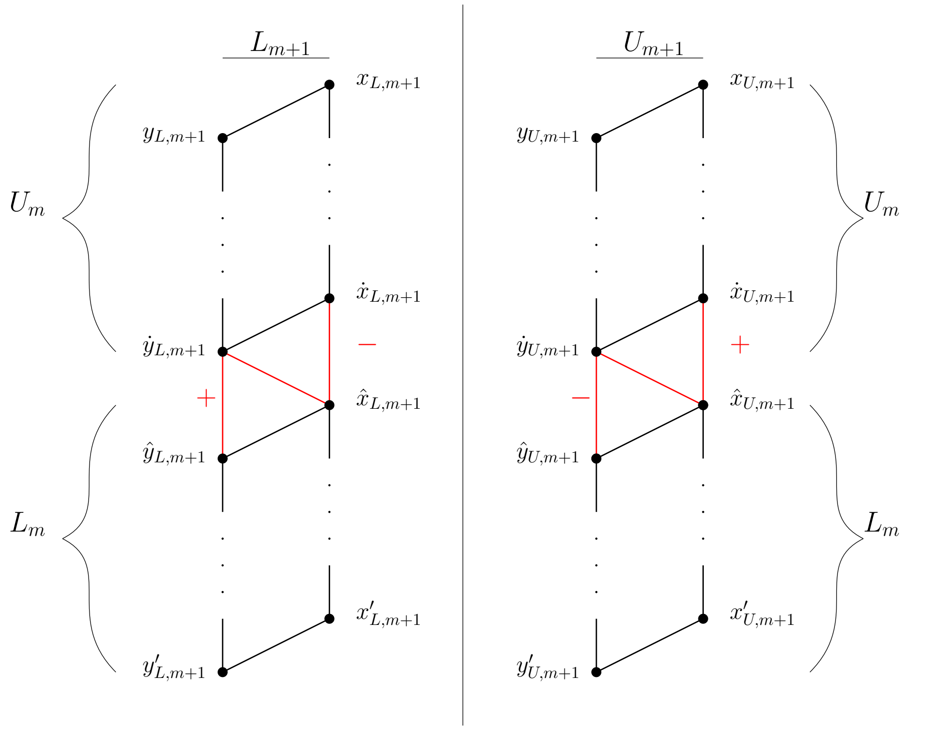

Figure 5-7

$$\dot{y}_{U,m+1} = t^{-2^m}\left(t^2 - S_{m+1}\right)x_{U,m+1} + t^{-2^m+2}S_{m+1}y_{U,m+1}.$$

Now replacing the expressions of $\dot{x}_{L,m+1}$ and $\dot{y}_{L,m+1}$ in Equation (5.3.10), we have:

$$\begin{aligned}
x'_{L,m+1} &= -t^{-2^m}S_{m+1}\left(t^{-2^m}\left(t^2\widetilde{S}_{m+1} - S_m^2\right)x_{L,m+1} + t^{-2^m+2}S_m^2 y_{L,m+1}\right)\\
&\quad + t^{-2^m}\left(S_{m+1} + S_m^2\right)\left(t^{-2^m}\left(t^2 - S_{m+1}\right)x_{L,m+1} + t^{-2^m+2}S_{m+1}y_{L,m+1}\right)\\
&= t^{-2^{m+1}}\left(-S_{m+1}\left(t^2\widetilde{S}_{m+1} - S_{m+1}^2\right) + \left(S_{m+1} + S_m^2\right)\left(t^2 - S_{m+1}\right)\right)x_{L,m+1}\\
&\quad + t^{-2^{m+1}+2}\Big(- S_{m+1}S_m^2 + S_{m+1}\left(S_{m+1} + S_m^2\right)\Big)y_{L,m+1}.
\end{aligned}$$

Now note that $S_{m+1}\widetilde{S}_{m+1} = \left(S_m + t^{2^{m-1}}\right)\left(S_m - t^{2^{m-1}}\right)$ and so $S_{m+1}\widetilde{S}_{m+1} = S_m^2 - t^{2^m}$. Thus

$$\begin{aligned}
x'_{L,m+1} &= t^{-2^{m+1}}\left(-t^2S_m^2 + t^{2^m+2} + S_{m+1}S_m^2 + t^2S_{m+1} - S_{m+1}^2 + t^2S_m^2 - S_m^2S_{m+1}\right)x_{L,m+1}\\
&\quad + t^{-2^{m+1}+2}\left(-S_{m+1}S_m^2 + S_{m+1}^2 + S_{m+1}S_m^2\right)y_{L,m+1}\\
&= t^{-2^{m+1}}\left(t^2S_{m+2} - S_{m+1}^2\right)x_{L,m+1} + t^{-2^{m+1}+2}S_{m+1}^2 y_{L,m+1}.
\end{aligned}$$

In a similar way, we get the other three equations below by replacing in Equations

(5.3.11), (5.3.12) and (5.3.13).

$$y'_{L,m+1} = t^{-2^{m+1}}\left(t^2 - \widetilde{S}_{m+2}\right)x_{L,m+1} + t^{-2^{m+1}+2}\widetilde{S}_{m+2}y_{L,m+1},$$

$$x'_{U,m+1} = t^{-2^{m+1}}\left(t^2\widetilde{S}_{m+2} - S^2_{m+1}\right)x_{U,m+1} + t^{-2^{m+1}+2}S^2_{m+1}y_{U,m+1},$$

$$y'_{U,m+1} = t^{-2^{m+1}}\left(t^2 - S_{m+2}\right)x_{U,m+1} + t^{-2^{m+1}+2}S_{m+2}y_{U,m+1}.$$

□

We need one more lemma before we can proceed to prove Theorem 5.3.1. We introduce a new element $\widetilde{a}_{m,2k} \in \mathbb{K}^2$ and show that it is also in $X_{m,2k}$. With the diagram U_m (instead of L_m), we write

$$\widetilde{a}_{m,2k} = \widetilde{a}_{m,2k}(1)X_{m,2k}(1) + \widetilde{a}_{m,2k}(2)X_{m,2k}(2), \text{ and}$$

$$\widetilde{b}_{m,2k} = \widetilde{b}_{m,2k}(1)X_{m,2k}(1) + \widetilde{b}_{m,2k}(2)X_{m,2k}(2)$$

with respect to the basis $X_{m,2k}(1), X_{m,2k}(2)$. We define them, for any $m \geqslant 2$, recursively as follows:

$$\widetilde{a}_{m,0}(1) := a_{m,0}(1), \widetilde{a}_{m,0}(2) := t^{-1}\left(\sum_{i=1}^{m-1} t^{2^i}\right), \widetilde{b}_{m,0}(1) := b_{m,0}(1), \widetilde{b}_{m,0}(2) := b_{m,0}(2),$$

$$\widetilde{a}_{m,2k} := t^{-1}\widetilde{a}_{m,2(k-1)} + \widetilde{b}_{m,2(k-1)} \text{ and } \widetilde{b}_{m,2k} := \widetilde{\delta}_{m,2(k-1)}\widetilde{a}_{m,2(k-1)}.$$

Where $\widetilde{\delta}_{m,2k}$ can be read from the diagram U_m (the equivalent of $\delta_{m,2k}$ for L_m). Note that since the top half of L_{m+1} as well as the top half of U_{m+1} are copies of U_m (by construction), we have the following equation:

$$\delta_{m+1,2k} = \widetilde{\delta}_{m+1,2k} = \widetilde{\delta}_{m,2k}, \text{ for all } 0 \leqslant k \leqslant 2^{m-1} - 1. \tag{5.3.14}$$

Lemma 5.3.5. *The following recursive formulas hold:*

$$\widetilde{a}_{m,2k}(2) = t^{-2}\widetilde{a}_{m,2(k-1)}(2) + t^{-1}\widetilde{b}_{m,2(k-1)}(2), \tag{5.3.15}$$
$$\widetilde{b}_{m,2k}(2) = \widetilde{\delta}_{m,2(k-1)}t^{-1}\widetilde{a}_{m,2(k-1)}(2). \tag{5.3.16}$$

Proof. Follows from the definitions of $\widetilde{a}_{m,2k}$ and $\widetilde{b}_{m,2k}$ and the formulas for $X_{m,2k}(1)$ and $X_{m,2k}(2)$ (from Lemma 5.3.3). □

We are now ready to prove the main theorem.

Proof of Theorem 5.3.1. We want to show that $\delta(x_{m,2k}, y) = 0$. We do this in seven steps. The first two steps will be used in proving the following two. In steps 1 and 2 we show that $a_{m,2k} \in X_{m,2k}$ and that $b_{m,2k} \in X_{m,2k}$. We then show in step 3 that these two elements are also in Y and thus showing that they are elements of $X_{m,2k} \cap Y$. In step 4, we show that $a_{m,2k}$ and $b_{m,2k}$ are linearly independent and hence $X_{m,2k} \cap Y$ contains a basis of $\mathbb{K}^2$. Finally, in step 5 we show that $tX_{m,2k} \cap Y = \{0\}$ which implies that $tX_{m,2k} \cap Y$ cannot contain a basis of $\mathbb{K}^2$ and so $d(x_{m,2k}, y) = \dim_{\Bbbk}(tX_{m,2k} \cap Y) = 0$.

Step 1. *We show that* $a_{m,2k} \in X_{m,2k}$.

Firstly, note that from Lemma 5.3.3 we have that $a_{m,2k}(1) = a_{m,2(k-1)}(1) + tb_{m,2(k-1)}(1) = a_{m,2(k-1)}(1) + \delta_{m,2(k-1)} t^2 a_{m,2(k-2)}(1)$ and that $a_{m,0}(1) = 1 + \left(\sum_{i=1}^{m-2} t^{2^i}\right) - t^{2^{m-1}} \in \Bbbk[t]$ and $a_{m,2}(1) = 1 + \left(\sum_{i=1}^{m-2} t^{2^i}\right) - t^{2^{m-1}} - t^2 \in \Bbbk[t]$. Therefore it is straightforward to see by induction on k that $a_{m,2k}(1) \in \Bbbk[t]$.

We then show that $a_{m,2k}(2) \in \Bbbk[t]$. We do this by showing that both $a_{m,2k}(2)$ and $\widetilde{a}_{m,2k}(2)$ are in $\Bbbk[t]$ by induction on m. This is straightforward to check for $m = 4$. Suppose then that $a_{m,2k}(2), \widetilde{a}_{m,2k}(2) \in \Bbbk[t]$ for all $0 \leqslant k \leqslant 2^{m-1} - 1$. We want to see that $a_{m+1,2k}(2), \widetilde{a}_{m+1,2k}(2) \in \Bbbk[t]$ for all $0 \leqslant k \leqslant 2^m - 1$. We start with $a_{m+1,2k}(2)$. We first show that $a_{m+1,2k}(2) \in \Bbbk[t]$ for $0 \leqslant k \leqslant 2^{m-1} - 1$, and then for $2^{m-1} \leqslant k \leqslant 2^m - 1$. Define $d_{m,2k} := a_{m+1,2k}(2) - \widetilde{a}_{m,2k}(2)$ for all $0 \leqslant k \leqslant 2^{m-1} - 1$. So $d_{m,2(k+2)} = \left(t^{-2} a_{m+1,2(k+1)}(2) + t^{-1} b_{m+1,2(k+1)}(2)\right) - \left(t^{-2} \widetilde{a}_{m,2(k+1)}(2) + t^{-1} \widetilde{b}_{m,2(k+1)}(2)\right)$, using the equations from Lemmas 5.3.3 and 5.3.5. Thus $d_{m,2(k+2)} = \left(t^{-2} a_{m+1,2(k+1)}(2) + \delta_{m+1,2k} t^{-2} a_{m+1,2k}(2)\right) - \left(t^{-2} \widetilde{a}_{m,2(k+1)}(2) + \widetilde{\delta}_{m,2k} t^{-2} \widetilde{a}_{m,2k}(2)\right)$. Now from Equation (5.3.14) we know that $\delta_{m+1,2k} = \widetilde{\delta}_{m,2k}$ and so $d_{m,2(k+2)} = t^{-2}\left(a_{m+1,2(k+1)}(2) - \widetilde{a}_{m,2(k+1)}(2)\right) + \delta_{m+1,2k} t^{-2}(a_{m+1,2k}(2) - \widetilde{a}_{m,2k}(2))$. Therefore $d_{m,2(k+2)} = t^{-2} d_{m,2(k+1)} + \delta_{m+1,2k} t^{-2} d_{m,2k}$.

Now note that $d_{m,0} = -t^{2^m-1}$ and $d_{m,2} = -t^{2^m-3}$, and so we conclude that for $0 \leqslant k \leqslant 2^{m-1} - 1$ the smallest power t can have as a term in $d_{m,2k}$ is greater or equal to $(2^m - 1) + (-2)(2^{m-1} - 1) = 1$. In other words $d_{m,k} \in \Bbbk[t]$. Therefore since $a_{m+1,2k}(2) = d_{m,2k} + \widetilde{a}_{m,2k}(2)$ and from the induction hypothesis we know that $\widetilde{a}_{m,2k}(2) \in \Bbbk[t]$ we can conclude that for $0 \leqslant k \leqslant 2^{m-1} - 1$, $a_{m+1,2k}(2) \in \Bbbk[t]$.

Then we show that $a_{m+1,2k}(2) \in \Bbbk[t]$ for $2^{m-1} \leqslant k \leqslant 2^m - 1$. Note that $a_{m+1,2(2^{m-1}-2)}(2) = \dot{x}_{L,m+1}$ and $a_{m+1,2(2^{m-1}-1)}(2) = \dot{y}_{L,m+1}$ in L_{m+1} from Figure 5-7 above, as long as we let $x_{L,m+1} = a_{m+1,0}(2)$ and $y_{L,m+1} = a_{m+1,2}(2)$. Now since the top half of L_{m+1} is a copy of U_m, Equations (5.3.4) and (5.3.5) from

Proposition 5.3.4 imply the following two equations:

$$a_{m+1,2(2^{m-1}-2)}(2) = t^{-2^m}\left(t^2\widetilde{S}_{m+1} - S_m^2\right)a_{m+1,0}(2) + t^{-2^m+2}S_m^2 a_{m+1,2}(2),$$

$$a_{m+1,2(2^{m-1}-1)}(2) = t^{-2^m}\left(t^2 - S_{m+1}\right)a_{m+1,0}(2) + t^{-2^m+2}S_{m+1}a_{m+1,2}(2).$$

Note that $\widetilde{S}_{m+1} = S_m - t^{2^{m-1}}$, $S_{m+1} = S_m + t^{2^{m-1}}$, $a_{m+1,0}(2) = t^{-1}\left(\sum_{i=1}^{m-1} t^{2^i}\right) - t^{2^m-1} = t^{-1}\left(\left(\sum_{i=1}^{m-1} t^{2^i}\right) - t^{2^m}\right) = t^{-1}\widetilde{S}_{m+2}$ and $a_{m+1,2}(2) = t^{-2}a_{m+1,0}(2) + t^{-1}b_{m+1,0}(2) = t^{-2}\left(t^{-1}\widetilde{S}_{m+2}\right) + t^{-1}(-1) = t^{-3}\widetilde{S}_{m+2} - t^{-1}$. Therefore, replacing we get:

$$\begin{aligned}
a_{m+1,2(2^{m-1}-2)}(2) &= t^{-2^m}\left(t^2\widetilde{S}_{m+1} - S_m^2\right)\left(t^{-1}S_{m+1} - t^{2^m-1}\right) \\
&\quad + t^{-2^m+2}S_m^2\left(t^{-3}\widetilde{S}_{m+2} - t^{-1}\right) \\
&= t^{-2^m+1}\widetilde{S}_{m+1}\widetilde{S}_{m+2} - t^{-2^m+1}S_m^2 \\
&= t^{-2^m+1}\left(S_m - t^{2^{m-1}}\right)\left(S_m + t^{2^{m-1}} - t^{2^m}\right) - t^{-2^m+1}S_m^2 \\
&= -t\left(S_m - t^{2m-1} + 1\right) \\
&= -t\left(\widetilde{S}_{m+1} + 1\right).
\end{aligned}$$

We also get:

$$\begin{aligned}
a_{m+1,2(2^{m-1}-1)}(2) &= t^{-2^m}\left(t^2 - S_{m+1}\right)\left(t^{-1}\widetilde{S}_{m+2}\right) \\
&\quad + t^{-2^m+2}S_{m+1}\left(t^{-3}\widetilde{S}_{m+2} - t^{-1}\right) \\
&= t^{-2^m+1}\left(\widetilde{S}_{m+2} - S_{m+1}\right) \\
&= t^{-2^m+1}\left(S_{m+1} - t^{2^m} - S_{m+1}\right) \\
&= -t.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
a_{m+1,2(2^{m-1})}(2) &= t^{-2}a_{m+1,2(2^{m-1}-1)}(2) + t^{-1}b_{m+1,2(2^{m-1}-1)}(2) \\
&= t^{-2}a_{m+1,2(2^{m-1}-1)}(2) + \delta_{m+1,2(2^{m-1}-2)}t^{-2}a_{m+1,2(2^{m-1}-2)}(2) \\
&= t^{-2}(-t) + (-1)\,t^{-2}\left(-t\left(\widetilde{S}_{m+1} + 1\right)\right) \\
&= t^{-1}\widetilde{S}_{m+1} \\
&= a_{m,0}(2).
\end{aligned}$$

We also have

$$\begin{aligned}
a_{m+1,2(2^{m-1}+1)}(2) &= t^{-2}a_{m+1,2(2^{m-1})}(2) + t^{-1}b_{m+1,2(2^{m-1})}(2)\\
&= t^{-2}a_{m+1,2(2^{m-1})}(2) + \delta_{m+1,2(2^{m-1}-1)}t^{-2}a_{m+1,2(2^{m-1}-1)}(2)\\
&= t^{-2}\left(a_{m,0}(2)\right) + (1)\, t^{-2}\left(-t\right)\\
&= t^{-2}\left(a_{m,0}(2)\right) + t^{-1}\left(-1\right)\\
&= t^{-2}\left(a_{m,0}(2)\right) + t^{-1}\left(b_{m,0}(2)\right)\\
&= a_{m,2}(2).
\end{aligned}$$

Now knowing that the bottom half of L_{m+1} is a copy of L_m and with the above equations (i.e. $a_{m+1,2(2^{m-1})}(2) = a_{m,0}(2)$ and $a_{m+1,2(2^{m-1}+1)}(2) = a_{m,2}(2)$) we conclude that $a_{m+1,2k}(2) = a_{m,2(k-2^{m-1})}(2)$ for all $2^{m-1} \leqslant k \leqslant 2^m - 1$ but by the induction hypothesis we know that $a_{m,2(k-2^{m-1})}(2) \in \Bbbk[t]$ and so $a_{m+1,2k}(2) \in \Bbbk[t]$ for all $2^{m-1} \leqslant k \leqslant 2^m - 1$.

The proof that $\widetilde{a}_{m+1,2k}(2) \in \Bbbk[t]$ for all $0 \leqslant k \leqslant 2^m - 1$ is similar. We start by showing that $\widetilde{a}_{m+1,2k}(2) \in \Bbbk[t]$ for all $0 \leqslant k \leqslant 2^{m-1} - 1$ and then for $2^{m-1} \leqslant k \leqslant 2^m - 1$. Define $\widetilde{d}_{m,2k} := \widetilde{a}_{m+1,2k}(2) - \widetilde{a}_{m,2k}(2)$ for all $0 \leqslant k \leqslant 2^{m-1} - 1$. So $\widetilde{d}_{m,2(k+2)} = \left(t^{-2}\widetilde{a}_{m+1,2(k+1)}(2) + t^{-1}\widetilde{b}_{m+1,2(k+1)}(2)\right) - \left(t^{-2}\widetilde{a}_{m,2(k+1)}(2) + t^{-1}\widetilde{b}_{m,2(k+1)}(2)\right)$, using the equations from Lemma 5.3.5. Thus $\widetilde{d}_{m,2(k+2)} = \left(t^{-2}\widetilde{a}_{m+1,2(k+1)}(2) + \widetilde{\delta}_{m+1,2k}t^{-2}\widetilde{a}_{m+1,2k}(2)\right) - \left(t^{-2}\widetilde{a}_{m,2(k+1)}(2) + \widetilde{\delta}_{m,2k}t^{-2}\widetilde{a}_{m,2k}(2)\right)$. Now from Equation (5.3.14) we know that $\widetilde{\delta}_{m+1,2k} = \widetilde{\delta}_{m,2k}$ and so $\widetilde{d}_{m,2(k+2)} = t^{-2}\left(\widetilde{a}_{m+1,2(k+1)}(2) - \widetilde{a}_{m,2(k+1)}(2)\right) + \widetilde{\delta}_{m+1,2k}t^{-2}(\widetilde{a}_{m+1,2k}(2) - \widetilde{a}_{m,2k}(2))$. Therefore $\widetilde{d}_{m,2(k+2)} = t^{-2}\widetilde{d}_{m,2(k+1)} + \widetilde{\delta}_{m+1,2k}t^{-2}\widetilde{d}_{m,2k}$.

Now note that, as previously, $\widetilde{d}_{m,0} = t^{2^m-1}$ and $\widetilde{d}_{m,2} = t^{2^m-3}$, and so we conclude that for $0 \leqslant k \leqslant 2^{m-1} - 1$ the smallest power t can have as a term in $\widetilde{d}_{m,2k}$ is greater or equal to $(2^m - 1) + (-2)(2^{m-1} - 1) = 1$. In other words $\widetilde{d}_{m,k} \in \Bbbk[t]$. Therefore since $\widetilde{a}_{m+1,2k}(2) = \widetilde{d}_{m,2k} + \widetilde{a}_{m,2k}(2)$ and from the induction hypothesis we know that $\widetilde{a}_{m,2k}(2) \in \Bbbk[t]$ we conclude that for $0 \leqslant k \leqslant 2^{m-1} - 1$, $a_{m+1,2k}(2) \in \Bbbk[t]$.

Finally, we want to show that $\widetilde{a}_{m+1,2k}(2) \in \Bbbk[t]$ for $2^{m-1} \leqslant k \leqslant 2^m - 1$. Note that $\widetilde{a}_{m+1,2(2^{m-1}-2)}(2) = \dot{x}_{U,m+1}$ and $\widetilde{a}_{m+1,2(2^{m-1}-1)}(2) = \dot{y}_{U,m+1}$ in U_{m+1} from Figure 5-7 above, as long as we let $x_{U,m+1} = \widetilde{a}_{m+1,0}(2)$ and $y_{U,m+1} = \widetilde{a}_{m+1,2}(2)$. Now since the top half of U_{m+1} is a copy of U_m, Equations (5.3.4) and (5.3.5) from Proposition 5.3.4 imply the following two equations:

$$\widetilde{a}_{m+1,2(2^{m-1}-2)}(2) = t^{-2^m}\left(t^2\widetilde{S}_{m+1} - S_m^2\right)\widetilde{a}_{m+1,0}(2) + t^{-2^m+2}S_m^2\widetilde{a}_{m+1,2}(2),$$

$$\widetilde{a}_{m+1,2(2^{m-1}-1)}(2) = t^{-2^m}\left(t^2 - S_{m+1}\right)\widetilde{a}_{m+1,0}(2) + t^{-2^m+2}S_{m+1}\widetilde{a}_{m+1,2}(2).$$

Note that $\widetilde{a}_{m+1,0}(2) = t^{-1}\left(\sum_{i=1}^{m} t^{2^i}\right) = t^{-1}S_{m+2}$ and $\widetilde{a}_{m+1,2}(2) = t^{-2}\widetilde{a}_{m+1,0}(2) + t^{-1}\widetilde{b}_{m+1,0}(2) = t^{-2}\left(t^{-1}S_{m+2}\right) + t^{-1}(-1) = t^{-3}S_{m+2} - t^{-1}$. Therefore, replacing we get:

$$\begin{aligned}
\widetilde{a}_{m+1,2(2^{m-1}-2)}(2) &= t^{-2^m}\left(t^2\widetilde{S}_{m+1} - S_m^2\right)\left(t^{-1}S_{m+2}\right) \\
&\quad + t^{-2^m+2}S_m^2\left(t^{-3}S_{m+2} - t^{-1}\right) \\
&= t^{-2^m+1}\widetilde{S}_{m+1}S_{m+2} - t^{-2^m+1}S_m^2 \\
&= t^{-2^m+1}\left(S_m - t^{2^{m-1}}\right)\left(S_m + t^{2^{m-1}} + t^{2^m}\right) - t^{-2^m+1}S_m^2 \\
&= t\left(S_m - t^{2^{m-1}} - 1\right) \\
&= t\left(\widetilde{S}_{m+1} - 1\right).
\end{aligned}$$

We also get:

$$\begin{aligned}
\widetilde{a}_{m+1,2(2^{m-1}-1)}(2) &= t^{-2^m}\left(t^2 - S_{m+1}\right)\left(t^{-1}S_{m+2}\right) \\
&\quad + t^{-2^m+2}S_{m+1}\left(t^{-3}S_{m+2} - t^{-1}\right) \\
&= t^{-2^m+1}S_{m+2} - t^{-2^m+1}S_{m+1} \\
&= t^{-2^m+1}\left(S_{m+1} + t^{2^m}\right) - t^{-2^m+1}S_{m+1} \\
&= t.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\widetilde{a}_{m+1,2(2^{m-1})}(2) &= t^{-2}\widetilde{a}_{m+1,2(2^{m-1}-1)}(2) + t^{-1}\widetilde{b}_{m+1,2(2^{m-1}-1)}(2) \\
&= t^{-2}\widetilde{a}_{m+1,2(2^{m-1}-1)}(2) + \widetilde{\delta}_{m+1,2(2^{m-1}-2)}t^{-2}\widetilde{a}_{m+1,2(2^{m-1}-2)}(2) \\
&= t^{-2}(t) + (1)\,t^{-2}\left(t\left(\widetilde{S}_{m+1} - 1\right)\right) \\
&= t^{-1}\widetilde{S}_{m+1} \\
&= a_{m,0}(2).
\end{aligned}$$

We also have

$$\begin{aligned}
\widetilde{a}_{m+1,2(2^{m-1}+1)}(2) &= t^{-2}\widetilde{a}_{m+1,2(2^{m-1})}(2) + t^{-1}\widetilde{b}_{m+1,2(2^{m-1})}(2) \\
&= t^{-2}\widetilde{a}_{m+1,2(2^{m-1})}(2) + \widetilde{\delta}_{m+1,2(2^{m-1}-1)}t^{-2}\widetilde{a}_{m+1,2(2^{m-1}-1)}(2) \\
&= t^{-2}\left(a_{m,0}(2)\right) + (-1)\, t^{-2}\,(t) \\
&= t^{-2}\left(a_{m,0}(2)\right) + t^{-1}\,(-1) \\
&= t^{-2}\left(a_{m,0}(2)\right) + t^{-1}\left(b_{m,0}(2)\right) \\
&= a_{m,2}(2).
\end{aligned}$$

Now knowing that the bottom half of U_{m+1} is a copy of L_m and with the above equations (i.e. $\widetilde{a}_{m+1,2(2^{m-1})}(2) = a_{m,0}(2)$ and $\widetilde{a}_{m+1,2(2^{m-1}+1)}(2) = a_{m,2}(2)$) we conclude that $\widetilde{a}_{m+1,2k}(2) = a_{m,2(k-2^{m-1})}(2)$ for all $2^{m-1} \leqslant k \leqslant 2^m - 1$ but by the induction hypothesis we know that $a_{m,2(k-2^{m-1})}(2) \in \Bbbk[t]$ and so $\widetilde{a}_{m+1,2k}(2) \in \Bbbk[t]$ for all $2^{m-1} \leqslant k \leqslant 2^m - 1$.

Step 2. *We show that* $b_{m,2k} \in X_{m,2k}$.

By definition $b_{m,2k} := \delta_{m,2(k-1)}a_{m,2(k-1)}$. Now step 1 above implies that $a_{m,2(k-1)} \in X_{m,2(k-1)}$. In other words, $a_{m,2(k-1)} = a_{m,2(k-1)}(1)X_{m,2(k-1)}(1) + a_{m,2(k-1)}(2)X_{m,2(k-1)}(2)$, with $a_{m,2(k-1)}(1), a_{m,2(k-1)}(2) \in \Bbbk[t]$.

Thus $b_{m,2k} = \delta_{m,2(k-1)}a_{m,2(k-1)}(1)X_{m,2(k-1)}(1) + \delta_{m,2(k-1)}a_{m,2(k-1)}(2)X_{m,2(k-1)}(2)$. And so using the equations from Lemma 5.3.3, we have:

$b_{m,2k} = \delta_{m,2(k-1)}a_{m,2(k-1)}(1)tX_{m,2k}(1) + \delta_{m,2(k-1)}a_{m,2(k-1)}(2)t^{-1}X_{m,2k}(2)$. Finally, the parity statement from the same lemma implies that $t^{-1}a_{m,2(k-1)}(2) \in \Bbbk[t]$, and so $b_{m,2k} \in X_{m,2k}$.

Step 3. *Both* $a_{m,2k}$ *and* $b_{m,2k}$ *are in* Y.

The proof is by induction on k. We start by explicitly computing $a_{m,0}$ and $b_{m,0}$. We have:

$$\begin{aligned}
a_{m,0} &= a_{m,0}(1)X_{m,0}(1) + a_{m,0}(2)X_{m,0}(2) \\
&= \left(1 + \left(\sum_{i=1}^{m-2} t^{2^i}\right) - t^{2^{m-1}}\right)\left(t^{-2^{m-1}}e_1 - t^{-(2^{m-1}+1)}\left(\sum_{i=1}^{m} t^{2^i}\right)e_2\right) \\
&+ \left(t^{-1}\left(\sum_{i=1}^{m-2} t^{2^i}\right) - t^{2^{m-1}-1}\right)\left(t^{2^{m-1}}e_2\right).
\end{aligned}$$

In order to simplify the exposition let us define as we did previously, $S_m :=$

$\sum_{i=1}^{m-2} t^{2^i}$. Thus:

$$\begin{aligned}
a_{m,0} &= t^{-2^{m-1}}\left(1+S_m-t^{2^{m-1}}\right)e_1 \\
&+\left(-t^{-(2^{m-1}+1)}S_{m+2}-t^{-(2^{m-1}+1)}S_mS_{m+2}+t^{-1}S_{m+2}\right)e_2 \\
&+\left(t^{2^{m-1}-1}S_m-t^{2^m-1}\right)e_2 \\
&= t^{-2^{m-1}}\left(1+S_m-t^{2^{m-1}}\right)e_1 \\
&+\Big(-t^{-(2^{m-1}+1)}\left(S_{m+1}+t^{2^m}\right)-t^{-(2^{m-1}+1)}S_m\left(S_m+t^{2^{m-1}}+t^{2^m}\right) \\
&+t^{-1}\left(S_m+t^{2^{m-1}}+t^{2^m}\right)+t^{2^{m-1}-1}S_m-t^{2^m-1}\Big)e_2 \\
&= t^{-2^{m-1}}\left(1+S_m-t^{2^{m-1}}\right)e_1 \\
&+\Big(-t^{-(2^{m-1}+1)}S_{m+1}-t^{2^{m-1}-1}-t^{-(2^{m-1}+1)}S_m^2-t^{-1}S_m-t^{2^{m-1}-1}S_m \\
&+t^{-1}S_m+t^{2^{m-1}-1}+t^{2^m-1}+t^{2^{m-1}-1}S_m-t^{2^m-1}\Big)e_2 \\
&= t^{-2^{m-1}}\left(1+S_m-t^{2^{m-1}}\right)e_1 \\
&+\left(-t^{-(2^{m-1}+1)}S_{m+1}-t^{-(2^{m-1}+1)}S_m^2\right)e_2.
\end{aligned}$$

So $a_{m,0}=t^{-2^{m-1}}\left(1+\left(\sum_{i=1}^{m-2}t^{2^i}\right)-t^{2^{m-1}}\right)e_1+\Big(-t^{-(2^{m-1}+1)}\left(\sum_{i=1}^{m-1}t^{2^i}\right)-t^{-(2^{m-1}+1)}\left(\sum_{i=1}^{m-2}t^{2^i}\right)^2\Big)e_2\in Y$. We also have:

$$\begin{aligned}
b_{m,0} &= b_{m,0}(1)X_{m,0}(1)+b_{m,0}(2)X_{m,0}(2) \\
&= -t\left(t^{-2^{m-1}}e_1-t^{-(2^{m-1}+1)}\left(\sum_{i=1}^{m}t^{2^i}\right)e_2\right)-\left(t^{2^{m-1}}e_2\right) \\
&= -t^{-2^{m-1}+1}e_1+\left(t^{-2^{m-1}}\left(t^{2^m}+\sum_{i=1}^{m-1}t^{2^i}\right)-t^{2^{m-1}}\right)e_2 \\
&= -t^{-2^{m-1}+1}e_1+\left(t^{2^{m-1}}+t^{-2^{m-1}}\left(\sum_{i=1}^{m-1}t^{2^i}\right)-t^{2^{m-1}}\right)e_2 \\
&= -t^{-2^{m-1}+1}e_1+t^{-2^{m-1}}\left(\sum_{i=1}^{m-1}t^{2^i}\right)e_2\in Y.
\end{aligned}$$

Now we assume that $a_{m,2(k-1)}$ and $b_{m,2(k-1)}$ are in Y. From the definitions, it is straightforward to see that $a_{m,2k},b_{m,2k}\in Y$.

Step 4. *The two elements $a_{m,2k}$ and $b_{m,2k}$ are linearly independent in $\mathbb{K}^2$.*

Firstly let $\alpha, \beta \in \mathbb{K}$ such that $\alpha a_{m,0} + \beta b_{m,0} = 0$. This means that

$\alpha\left(a_{m,0}(1)X_{m,0}(1) + a_{m,0}(2)X_{m,0}(2)\right) + \beta\left(b_{m,0}(1)X_{m,0}(1) + b_{m,0}(2)X_{m,0}(2)\right) = 0$. So $\left(\alpha a_{m,0}(1) + \beta b_{m,0}(1)\right)X_{m,0}(1) + \left(\alpha a_{m,0}(2) + \beta b_{m,0}(2)\right)X_{m,0}(2) = 0$.

Now $X_{m,0}(1)$ and $X_{m,0}(2)$ are linearly independent in $\mathbb{K}^2$ so we get that

$$\alpha a_{m,0}(1) + \beta b_{m,0}(1) = \alpha\left(-1 + \left(\sum_{i=1}^{m-2} t^{2^i}\right) - t^{2^{m-1}}\right) - t\beta = 0, \text{ and}$$

$$\alpha a_{m,0}(2) + \beta b_{m,0}(2) = \alpha\left(t^{-1}\left(\sum_{i=1}^{m-2} t^{2^i}\right) - t^{2^{m-1}-1}\right) - \beta = 0.$$

Therefore $\alpha = \beta = 0$ showing that $a_{m,0}$ and $b_{m,0}$ are linearly independent.

Let us now assume that $a_{m,2(k-1)}$ and $b_{m,2(k-1)}$ are linearly independent and prove that $a_{m,2k}$ and $b_{m,2k}$ are also linearly independent in $\mathbb{K}^2$. For this, let $\alpha, \beta \in \mathbb{K}$ and assume that $\alpha a_{m,2k} + \beta b_{m,2k} = 0$. By definition $a_{m,2k} = a_{m,2k}(1)X_{m,2k}(1) + a_{m,2k}(2)X_{m,2k}(2)$ and $b_{m,2k} = b_{m,2k}(1)X_{m,2k}(1) + b_{m,2k}(2)X_{m,2k}(2)$, and since $X_{m,2k}(1) = t^{-1}X_{m,2(k-1)}(1)$ and $X_{m,2k}(2) = tX_{m,2(k-1)}(2)$, we get that:

$\alpha\left(a_{m,2k}(1)\left(t^{-1}X_{m,2(k-1)}(1)\right) + a_{m,2k}(2)\left(tX_{m,2(k-1)}(2)\right)\right) +$
$\beta\left(b_{m,2k}(1)\left(t^{-1}X_{m,2(k-1)}(1)\right) + b_{m,2k}(2)\left(tX_{m,2(k-1)}(2)\right)\right) = 0.$

But then using the equations from Lemma 5.3.3, we have:

$\left(t^{-1}\alpha + s_{m,2(k-1)}\beta\right)\left(a_{m,2(k-1)}(1)X_{m,2(k-1)}(1) + a_{m,2(k-1)}(2)X_{m,2(k-1)}(2)\right) +$
$\alpha\left(b_{m,2(k-1)}(1)X_{m,2(k-1)}(1) + b_{m,2(k-1)}(2)X_{m,2(k-1)}(2)\right) = 0.$

We also have by definition that

$a_{m,2(k-1)} = a_{m,2(k-1)}(1)X_{m,2(k-1)}(1) + a_{m,2(k-1)}(2)X_{m,2(k-1)}(2)$ and $b_{m,2(k-1)} = b_{m,2(k-1)}(1)X_{m,2(k-1)}(1) + b_{m,2(k-1)}(2)X_{m,2(k-1)}(2)$, and so we have:

$\left(t^{-1}\alpha + s_{m,2(k-1)}\beta\right)a_{m,2(k-1)} + \alpha b_{m,2(k-1)} = 0.$

Finally the fact that $a_{m,2(k-1)}$ and $b_{m,2(k-1)}$ are linearly independent implies that $\alpha = \beta = 0$. Therefore $a_{m,2k}$ and $b_{m,2k}$ are indeed linearly independent in $\mathbb{K}^2$.

Step 5. *Proving that $tX_{m,2k} \cap Y = \{0\}$ for all $m \geqslant 2$ and $0 \leqslant k \leqslant 2^{m-1} - 1$.*

Let $f \in tX_{m,2k} \cap Y$. Then there are $\alpha, \beta \in \Bbbk[t]$ such that $f = \alpha tX_{m,2k}(1) + \beta tX_{m,2k}(2)$. Thus we have that

$f = \alpha t\left(t^{-(2^{m-1}+k)}e_1 - t^{-(2^{m-1}+k+1)}\left(\sum_{i=1}^m t^{2^i}\right)e_2\right) + \beta t\left(t^{2^{m-1}+k}e_2\right)$. And so multiplying by $t^{2^{m-1}+k}$, we have that: $t^{2^{m-1}+k}f = (\alpha t)e_1 + \left(\beta t^{2^m+2k+1} - \alpha\sum_{i=1}^m t^{2^i}\right)e_2$.

Since $f \in Y$, the highest power of t in α is at most $2^{m-1} + k - 1$ and the highest power of t in $D := \beta t^{2^m+2k+1} - \alpha\sum_{i=1}^m t^{2^i}$ is at most $2^{m-1} + k$.

Terms of degree $2^m + 2k + 1$ and above determine β uniquely in terms of α.

The terms in D that are of degrees $2^{m-1}+k+1, \dots, 2^m+2k$ must be zero. There are $2^{m-1}+k$ of these terms and they come only from α. Now α is in $\Bbbk^{2^{m-1}+k}$ and we can look at the linear map $\Bbbk^{2^{m-1}+k} \to \Bbbk^{2^{m-1}+k}$ sending α to the set of these coefficients (that are all equal to zero). It is not difficult to see that this map is surjective, hence injective by rank-nullity, so $\alpha = 0$ and $\beta = 0$. $\square$

5.3.2 Proof of Theorem 5.3.2

Without loss of generality, let us suppose that $g_y(0) = y_1$.

By definition, $x_{0,0} = [\![e_1, te_2]\!]$ and so $x_{0,0} = [\![t^{-1}e_1, e_2]\!]$. Let $X_{0,0} := \Bbbk[t]t^{-1}e_1 \oplus \Bbbk[t]e_2$, so that $x_{0,0} = [X_{0,0}]$ and note that $X_{0,0} \cap Y_1 = \Bbbk t^{-1}e_1 \oplus \Bbbk e_1 \oplus \Bbbk te_1 \oplus \Bbbk e_2$, which contains a basis for $\mathbb{K}^2$ while $tX_{0,0} \cap Y_1 = \Bbbk e_1 \oplus \Bbbk te_1$ which does not contain such a basis for $\mathbb{K}^2$. This means that $d(x_{0,0}, y_1) = \dim_{\Bbbk}(tX_{0,0} \cap Y_1) = 2$. And so $\Pi_y(x_{0,0}) = y_1$.

Now let $\mathscr{A}_x : \mathbb{Z} \to A_x$ be the labelled apartment such that $\mathscr{A}_x(0) := x_{0,0}$ and $\mathscr{A}_x(1) := x_{1,1}$. Clearly, its associated apartment is A_x.

In the following proposition we show that for all $m \geqslant 2$ and $0 \leqslant l \leqslant 2^{m-2}-1$ the codistances $d(x_{m,4l+1}, y_1) = 2$ and $d(x_{m,4l+3}, y_2) = 2$. So in other words $\Pi_y(x_{m,4l+1}) = y_1$ and $\Pi_y(x_{m,4l+3}) = y_2$, therefore we can conclude that $\Pi_y \circ \mathscr{A}_x = g_y$, showing that the apartment A_x parahorically projects onto g_y from y, and thus proving Theorem 5.3.2.

Proposition 5.3.6. *We have that $d(x_{m,4l+1}, y_1) = 2$ and $d(x_{m,4l+3}, y_2) = 2$ for all $m \geqslant 2$ and $0 \leqslant l \leqslant 2^{m-2}-1$.*

Let $X_{m,4l+1} = [X_{m,4l+1}(1), X_{m,4l+1}(2)]$ be a $\Bbbk[t]$-lattice in $\mathbb{K}^2$ where $X_{m,4l+1}(1) := t^{-(2^{m-1}+2l-1)}e_1 - t^{-(2^{m-1}+2l)}\left(\sum_{i=1}^m t^{2^i}\right)e_2$ and $X_{m,4l+1}(2) := t^{2^{m-1}+2l}e_2$. Thus $x_{m,4l+1} = [X_{m,4l+1}]$. Similarly, let $X_{m,4l+3} = [X_{m,4l+3}(1), X_{m,4l+3}(2)]$ be a $\Bbbk[t]$-lattice in $\mathbb{K}^2$ where $X_{m,4l+3}(1) := t^{-(2^{m-1}+2l+2)}e_1 - t^{-(2^{m-1}+2l+3)}\left(\sum_{i=1}^m t^{2^i}\right)e_2$ and $X_{m,4l+3}(2) := t^{2^{m-1}+2l+1}e_2$. Thus $x_{m,4l+3} = [X_{m,4l+3}]$.

Lemma 5.3.7. *With the above definitions, we have $tX_{m,4l+1} \subseteq X_{m,4l}$ and $tX_{m,4l+3} \subseteq X_{m,4l+2}$. And so $tX_{m,4l+1} \cap Y$ and $tX_{m,4l+3} \cap Y$ cannot contain a basis for $\mathbb{K}^2$.*

Proof. From the definitions, we have

$$tX_{m,4l+1} = [X_{m,4l}(1), tX_{m,4l}(2)] \subseteq [X_{m,4l}(1), X_{m,4l}(2)] = X_{m,4l}$$

and $tX_{m,4l+3} = [X_{m,4l+2}(1), tX_{m,4l+2}(2)] \subseteq [X_{m,4l+2}(1), X_{m,4l+2}(2)] = X_{m,4l+2}$.

Now suppose that $tX_{m,4l+1} \cap Y$ contains a basis for $\mathbb{K}^2$. Since $d(x_{m,4l+1}, y) = 1$ then $tX_{m,4l+1} \cap Y$ must contain at least 3 $\Bbbk$-linearly independent elements, but then those elements would also be in $X_{m,4l} \cap Y$ and from the proof of Theorem 5.3.1, we know that $d(x_{m,4l}, y) = 0 = \dim_{\Bbbk}(X_{m,4l} \cap Y) - 2 \geqslant 3 - 2 = 1$ leading to a contradiction. A similar argument holds for $tX_{m,4l+3} \cap Y$. $\square$

Lemma 5.3.8. *We have that* $\dim_{\Bbbk}(tX_{m,4l+1} \cap Y) = 1$*. Similarly,* $\dim_{\Bbbk}(tX_{m,4l+3} \cap Y) = 1$.

Proof. Firstly, note that $X_{m,4l+1} \cap Y$ contains a basis for $\mathbb{K}^2$ since $X_{m,4l} \subseteq X_{m,4l+1}$ and from the proof of Theorem 5.3.1 we know that $X_{m,4l} \cap Y$ contains a basis for $\mathbb{K}^2$. And from Lemma 5.3.7 we know that $tX_{m,4l+1} \cap Y$ does not contain a basis for $\mathbb{K}^2$. Therefore, $d(x_{m,4l+1}, y) = \dim_{\Bbbk}(tX_{m,4l+1} \cap Y)$, but from Theorem 5.3.1 we know that $d(x_{m,4l+1}, y) = 1$ and so $\dim_{\Bbbk}(tX_{m,4l+1} \cap Y) = 1$. The proof for $tX_{m,4l+3} \cap Y$ is done similarly. $\square$

Lemma 5.3.9. *We have that* $X_{m,4l+1}(1) = f_o(t)e_1 + g_e(t)e_2$ *and* $X_{m,4l+1}(2) = \tilde{g}_e(t)e_2$ *where* f_o *is a Laurent polynomial over* $\Bbbk$ *with only odd powers of* t*, while* g_e *and* $\tilde{g}_e$ *are Laurent polynomials over* $\Bbbk$ *with only even powers of* t*.*

We also have $X_{m,4l+3}(1) = p_e(t)e_1 + q_o(t)e_2$ *and* $X_{m,4l+3}(2) = \tilde{q}_o(t)e_2$ *where* p_e *is a Laurent polynomial over* $\Bbbk$ *with only even powers of* t*, while* q_o *and* $\tilde{q}_o$ *are Laurent polynomials over* $\Bbbk$ *with only odd powers of* t*.*

Proof. This follows by definitions. $\square$

Lemma 5.3.10. *Let* $a, b \in X_{m,4l+1} \cap Y$ *such that over* $\Bbbk$ *they are linearly independent. Then it is not possible to write* $a = f_e(t)e_1 + g_o(t)e_2$ *and* $b = p_e(t)e_1 + q_o(t)e_2$ *where* f_e *and* p_e *are Laurent polynomials over* $\Bbbk$ *with all powers of* t *being even while* g_o *and* q_o *are Laurent polynomials over* $\Bbbk$ *with odd powers of* t*.*

Similarly, let $\tilde{a}, \tilde{b} \in X_{m,4l+3} \cap Y$ *such that over* $\Bbbk$ *they are linearly independent. Then it is not possible to write* $\tilde{a} = f_o(t)e_1 + g_e(t)e_2$ *and* $\tilde{b} = p_o(t)e_1 + q_e(t)e_2$ *where* f_o *and* p_o *are Laurent polynomials over* $\Bbbk$ *with all powers of* t *being odd while* g_e *and* q_e *are Laurent polynomials over* $\Bbbk$ *with even powers of* t*.*

Proof. Suppose that $a = f_e(t)e_1 + g_o(t)e_2$ and $b = p_e(t)e_1 + q_o(t)e_2$, as described in the lemma. We have that $a \in X_{m,4l+1}$ so $a = f_e(t)e_1 + g_o(t)e_2 = \alpha(t)X_{m,4l+1}(1) + \beta(t)X_{m,4l+1}(2)$, where $\alpha, \beta \in \Bbbk[t]$. Now Lemma 5.3.9 implies that for this equation to hold, powers of t in both α and β must all be odd. But then this implies that $t^{-1}\alpha, t^{-1}\beta \in \Bbbk[t]$ which in turns implies that $t^{-1}a \in X_{m,4l+1}$. Now, $t^{-1}a$

is obviously also in Y, thus it is in $X_{m,4l+1} \cap Y$. This same argument holds for b and $t^{-1}b$. Finally since a and b are $\Bbbk$-linearly independent, we get that $\dim_{\Bbbk}(X_{m,4l+1} \cap Y) \geqslant 4$, but from Lemma 5.3.7 we know that $tX_{m,4l+1} \cap Y$ cannot contain a basis for $\mathbb{K}^2$ and so $d(x_{m,4l+1}, y) = 1 \geqslant \dim_{\Bbbk}(X_{m,4l+1} \cap Y) - 2 \geqslant 4 - 2 = 2$ leading to a contradiction. A similar argument holds for $\tilde{a}, \tilde{b} \in X_{m,4l+3} \cap Y$. $\square$

Let $a = p(t)e_1 + q(t)e_2$, with Laurent polynomials p and q over $\Bbbk$. Let us now write $p(t) = p_e(t) + p_o(t)$ and $q(t) = q_e(t) + q_o(t)$ where p_e and q_e are the parts of p and q respectively that have only powers of t that are even, and similarly p_o and q_o are the parts of p and q respectively that have only powers of t that are odd. Finally, let $a_{e,o} := p_e(t)e_1 + q_o(t)e_2$ and $a_{o,e} := p_o(t)e_1 + q_e(t)e_2$ so that $a = a_{e,o} + a_{o,e}$.

Lemma 5.3.11. *With the notation above, $a \in tX_{m,4l+1} \cap Y_1$ implies that $a_{e,o}, a_{o,e} \in tX_{m,4l+1} \cap Y_1$. Similarly, $a \in tX_{m,4l+3} \cap Y_1$ implies that $a_{e,o}, a_{o,e} \in tX_{m,4l+3} \cap Y_1$.*

Proof. The case of $tX_{m,4l+3} \cap Y_1$ is similar to $tX_{m,4l+1} \cap Y_1$, so we will only prove the first claim of the lemma. A similar argument proves the second claim.

Let $a = p(t)e_1 + q(t)e_2$ for some Laurent polynomials p and q over $\Bbbk$. We have $a \in Y_1$, so $p \in t\Bbbk[t^{-1}]$ and $q \in \Bbbk[t^{-1}]$ which implies that $p_e, p_o \in t\Bbbk[t^{-1}]$ and $q_e, q_o \in \Bbbk[t^{-1}]$. Therefore $a_{e,o}, a_{o,e} \in Y_1$.

Now $a \in tX_{m,4l+1}$, i.e. $a = \alpha(t)tX_{m,4l+1}(1) + \beta(t)tX_{m,4l+1}(2)$ for some polynomials $\alpha, \beta \in \Bbbk[t]$. With notations from Lemma 5.3.9, we have $X_{m,4l+1}(1) = f_o(t)e_1 + g_e(t)e_2$ and $X_{m,4l+1}(2) = \tilde{g}_e(t)e_2$, and so $a = \alpha(t)t\,(f_o(t)e_1 + g_e(t)e_2) + \beta(t)t\,(\tilde{g}_e(t)e_2) = (\alpha(t)tf_o(t))\,e_1 + (\alpha(t)tg_e(t) + \beta(t)t\tilde{g}_e(t))\,e_2$.

Let us now write $\alpha = \alpha_e + \alpha_o$ and $\beta = \beta_e + \beta_o$ where α_e and β_e are the parts of α and β respectively that have powers of t that are even, and similarly α_o and β_o are the parts with odd powers of t. So

$$\begin{aligned} a = {} & (\alpha_e(t)tf_o(t))\,e_1 + (\alpha_e(t)tg_e(t) + \beta_e(t)t\tilde{g}_e(t))\,e_2 \\ & + (\alpha_o(t)tf_o(t))\,e_1 + (\alpha_o(t)tg_e(t) + \beta_o(t)t\tilde{g}_e(t))\,e_2. \end{aligned}$$

Note that the powers of t in both $\alpha_e(t)tf_o(t)$ and $\alpha_o(t)tg_e(t) + \beta_o(t)t\tilde{g}_e(t)$ are all even, while the powers of t in $\alpha_e(t)tg_e(t) + \beta_e(t)t\tilde{g}_e(t)$ and $\alpha_o(t)tf_o(t)$ are odd.

Thus, $a_{e,o} = (\alpha_e(t)tf_o(t))\,e_1 + (\alpha_e(t)tg_e(t) + \beta_e(t)t\tilde{g}_e(t))\,e_2$ and $a_{o,e} = (\alpha_o(t)tf_o(t))\,e_1 + (\alpha_o(t)tg_e(t) + \beta_o(t)t\tilde{g}_e(t))\,e_2$. But now $a_{e,o} = \alpha_e(t)t\,(f_o(t)e_1 + g_e(t)e_2) + \beta_e(t)t\,(\tilde{g}_e(t)e_2) =$

$\alpha_e(t)tX_{m,4l+1}(1)+\beta_e(t)tX_{m,4l+1}(2) \in tX_{m,4l+1}$ and $a_{o,e} = \alpha_o(t)t\left(f_o(t)e_1 + g_e(t)e_2\right)+ \beta_o(t)t\left(\tilde{g}_e(t)e_2\right) = \alpha_o(t)tX_{m,4l+1}(1) + \beta_o(t)tX_{m,4l+1}(2) \in tX_{m,4l+1}$. □

Lemma 5.3.12. *Let $a \in tX_{m,4l+1} \cap Y$. Then $a = p_e(t)e_1 + q_o(t)e_2$, for some Laurent polynomials p_e and q_o over $\mathbb{k}$ where there are only even powers of t in $p_e(t)$ and only odd powers of t in $q_o(t)$.*

Similarly, if $a \in tX_{m,4l+3} \cap Y$. Then $a = p_o(t)e_1 + q_e(t)e_2$, for some Laurent polynomials p_o and q_e over $\mathbb{k}$ where there are only odd powers of t in $p_o(t)$ and only even powers of t in $q_e(t)$.

Proof. Let $a \in tX_{m,4l+1} \cap Y$. Then since $Y \subseteq Y_1$ and from Lemmas 5.3.11 and 5.3.8, we must have either $a = p_e(t)e_1 + q_o(t)e_2$ or $a = p_o(t)e_1 + q_e(t)e_2$, for some Laurent polynomials p_e, p_o, q_e, q_o over $\mathbb{k}$ where there are only even powers of t in $p_e(t)$ and $q_e(t)$ and only odd powers of t in $p_o(t)$ and $q_o(t)$.

Suppose that $a = p_o(t)e_1 + q_e(t)e_2$, then since $a \in tX_{m,4l+1}$ there exist polynomials α, β over $\mathbb{k}$ such that $a = \alpha(t)tX_{m,4l+1}(1) + \beta(t)tX_{m,4l+1}(2)$.

As in the proof of Lemma 5.3.11, replacing $X_{m,4l+1}(1)$ and $X_{m,4l+1}(2)$ with $f_o(t)e_1 + g_e(t)e_2$ and $\tilde{g}_e(t)e_2$ respectively, we get:

$$a = \left(\alpha(t)tf_o(t)\right)e_1 + \left(\alpha(t)tg_e(t) + \beta(t)t\tilde{g}_e(t)\right)e_2.$$

This implies that $\alpha(t)$ must have all powers of t to be odd, and thus all powers of t in $\beta(t)$ must be odd as well. And so we have $\alpha(t) = t\tilde{\alpha}(t)$ and $\beta(t) = t\tilde{\beta}(t)$ with $\tilde{\alpha}, \tilde{\alpha} \in \mathbb{k}[t]$.

Note now that $t^{-1}a = \alpha(t)X_{m,4l+1}(1) + \beta(t)X_{m,4l+1}(2) = \tilde{\alpha}(t)tX_{m,4l+1}(1) + \tilde{\beta}(t)tX_{m,4l+1}(2) \in tX_{m,4l+1}$. Of course $t^{-1}a \in Y$, and so $t^{-1}a \in tX_{m,4l+1} \cap Y$ which is a contradiction since we know from Lemma 5.3.8 that $\dim_{\mathbb{k}}(tX_{m,4l+1} \cap Y) = 1$.

The case of $tX_{m,4l+3} \cap Y$ is done similarly. □

We are now ready to prove Proposition 5.3.6 and hence proving Theorem 5.3.2.

Proof of Proposition 5.3.6. Proving $d(x_{m,4l+3}, y_2) = 2$ is done similarly to proving that $d(x_{m,4l+1}, y_1) = 2$, and so we focus on the latter. We first show in Step 1 that $X_{m,4l+1} \cap Y_1$ contains a basis for $\mathbb{K}^2$ and in Step 2 we show that $tX_{m,4l+1} \cap Y_1$ does not contain such a basis for $\mathbb{K}^2$. This implies that $d(x_{m,4l+1}, y_1) = \dim_{\mathbb{k}}(tX_{m,4l+1} \cap Y_1)$. But note from Lemma 5.3.8 that $\dim_{\mathbb{k}}(tX_{m,4l+1} \cap Y) = 1$ and since $tX_{m,4l+1} \cap Y \subseteq tX_{m,4l+1} \cap Y_1$ we must have that $d(x_{m,4l+1}, y_1) \geqslant 1$. Finally from the definition of codistance and Theorem 5.3.1, we know that $d(x_{m,4l+1}, y_1)$ must be equal to either 0 or 2. Therefore, $d(x_{m,4l+1}, y_1) = 2$.

Step 1. *We show that $X_{m,4l+1} \cap Y_1$ contains a basis for $\mathbb{K}^2$.*

This is straightforward since $X_{m,4l} \subseteq X_{m,4l+1}$, $Y \subseteq Y_1$ so $X_{m,4l} \cap Y \subseteq X_{m,4l} \cap Y_1$, and from the proof of Theorem 5.3.2 we know that $X_{m,4l} \cap Y$ contains a basis of $\mathbb{K}^2$.

Step 2. *We show that $tX_{m,4l+1} \cap Y_1$ does not contain a basis for $\mathbb{K}^2$.*

We prove this by contradiction, so suppose that $tX_{m,4l+1} \cap Y_1$ contains a basis for $\mathbb{K}^2$.

From Lemma 5.3.8 we know that $\dim_{\Bbbk}(tX_{m,4l+1} \cap Y) = 1$ and so let a be a non-zero element in $tX_{m,4l+1} \cap Y \subseteq tX_{m,4l+1} \cap Y_1$. Now from Lemma 5.3.12, we also know that $a = p_e(t)e_1 + q_o(t)e_2$, for some Laurent polynomials p_e and q_o in $\Bbbk[t^{-1}]$ where there are only even powers of t in $p_e(t)$ and only odd powers of t in $q_o(t)$. Thus $tp_e \in t\Bbbk[t^{-1}]$ but $tq_o \in \Bbbk[t^{-1}]$, and so $ta \in tX_{m,4l+1} \cap Y_1$.

Now since a and ta are not $\mathbb{K}$-linearly independent, and we are assuming that $tX_{m,4l+1} \cap Y_1$ contains a basis for $\mathbb{K}^2$, we must have another element in $tX_{m,4l+1} \cap Y_1$, call it b, that together with ta forms a basis for $\mathbb{K}^2$.

Let $b = r(t)e_1 + s(t)e_2$, with Laurent polynomials r and s over $\Bbbk$. Let us now write $r(t) = r_e(t) + r_o(t)$ and $s(t) = s_e(t) + s_o(t)$ where r_e and s_e are the parts of r and s respectively that have only powers of t that are even, and similarly r_o and s_o are the parts of r and s respectively that have only powers of t that are odd. Finally, as we did before, let $b_{e,o} := r_e(t)e_1 + s_o(t)e_2$ and $b_{o,e} := r_o(t)e_1 + s_e(t)e_2$ so that $b = b_{e,o} + b_{o,e}$.

Lemma 5.3.11 implies that $b_{e,o}, b_{o,e} \in tX_{m,4l+1} \cap Y_1$. Note that since the term of e_1 in $b_{e,a}$ has only even powers of t, we have that $b_{e,a}$ is also in Y and so in $tX_{m,4l+1} \cap Y$. Therefore Lemma 5.3.8 implies that a and $b_{e,a}$ cannot be $\mathbb{K}$-linearly independent, which in turn implies that a and $b_{o,e}$ must be $\mathbb{K}$-linearly independent. Thus, a and $t^{-1}b_{o,e}$ are $\mathbb{K}$-linearly independent, but we know that $a, t^{-1}b_{o,e} \in X_{m,4l+1} \cap Y$ (since $ta, b_{o,e} \in tX_{m,4l+1} \cap Y_1$) and both a and $t^{-1}b_{o,e}$ have e_1 terms which have only even powers of t and e_2 terms which have only odd powers of t, contradiction Lemma 5.3.10. $\square$

5.4 Further work

In this thesis, we specialised to the case of $\widetilde{A_1}$. After noting that the incidence system of the parahoric building of type $\widetilde{A_1}$ is a tree, we introduced the notion of parahoric projection (Definition 5.2.1) in the context of trees, then under certain circumstances (i.e. geometrical configurations being without repetition) we solved the lifting problem (Theorem 5.2.3). And as a corollary we obtained the existence of weakly opposite apartments in $\mathcal{T}_0$ to a fixed vertex in the tree $\mathcal{T}_\infty$ (Corollary 5.2.4). Finally as an illustration we constructed a geometrical configuration g_y and an apartment A_x in $\mathcal{T}_0$: we showed that A_x is a weakly opposite apartment to some fixed y in $\mathcal{T}_\infty$ (Theorem 5.3.1), and that it is in fact a lift for the geometrical configuration g_y (Theorem 5.3.2).

Note that all of this was possible since trees have a convenient structure that allowed us to define a distance map d, allowing us among other things to use the definition of weakly opposite apartments for trees (Definition 5.1.1). Of course this is not true in the general case: for instance recall that the associated incidence system to the building of type $\widetilde{A_2}$ is not a tree (see Figure 5-8 below).

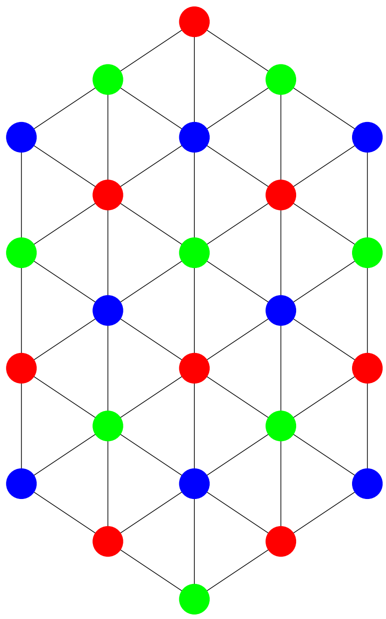

Figure 5-8

A generalised definition for weakly opposite apartments in twin buildings should involve residues, since now we have a distance δ only on the buildings and not on their incidence systems.

Definition 5.4.1 (Weakly opposite apartments)**.** Let $(\mathcal{B}_0, \mathcal{B}_\infty, d)$ be a twin tree and fix an i-residue R in $\mathcal{B}_\infty$. We say that the apartment A in $\mathcal{B}_0$ is *weakly opposite* to R if for all chambers $a \in A$ and $b \in R$, we have that $\delta(a, b)$ has either length 0 or 1 as a word in W.

Future work, which is beyond the scope of this thesis, could involve finding a weakly opposite apartment in the building of type $\widetilde{A_2}$ with the above definition. It could also involve working out a generalisation of parahoric projection onto residues of $\widetilde{A_2}$.

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